One-Step Calculation of Translational and Rotational Kinetic Energy Stored in a Rolling Body as a Result of an Impulsive Force

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Outline:

- Introduction.
- Definition of impulse and unit step functions.
- Impulse-unit step relationship.
- Special singularity integral and its derivation.
- Example: One-step calculation of total kinetic energy stored in a rolling body due to an impulsive force.
- Conclusion.

Introduction:

- Extension of previous one-step calculation of kinetic energy of simple translational moving body (Osterberg/Inan, CANCAM 2007) to more general rolling body (both translational and rotational).

- Kinetic energy stored in rolling body (translational plus rotational):

$$W=(1/2)mv^2 + (1/2)I\omega^2$$

- Calculation of kinetic energy due to a ***continuous*** force is well-known in the literature.

- Calculation of kinetic energy due to a ***discontinuous*** force is not addressed in the literature, but should be.

Introduction:

- Authors address the ***discontinuous*** force case by incorporating a special singularity integral.
- This special singularity integral involves the well-known impulse and the unit step functions.
- A simple example of the calculation of the kinetic energy due to a ***discontinuous impulsive*** force using this special integral is presented.

Impulse Function:

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0$$

and

$$\int_{-\infty}^{\infty} \delta(t)\,dt = \int_{0^-}^{0^+} \delta(t)\,dt = 1$$

$$\delta(0)\ \text{is undefined}$$

Unit Step Function:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

$$u(0) \text{ is undefined}$$

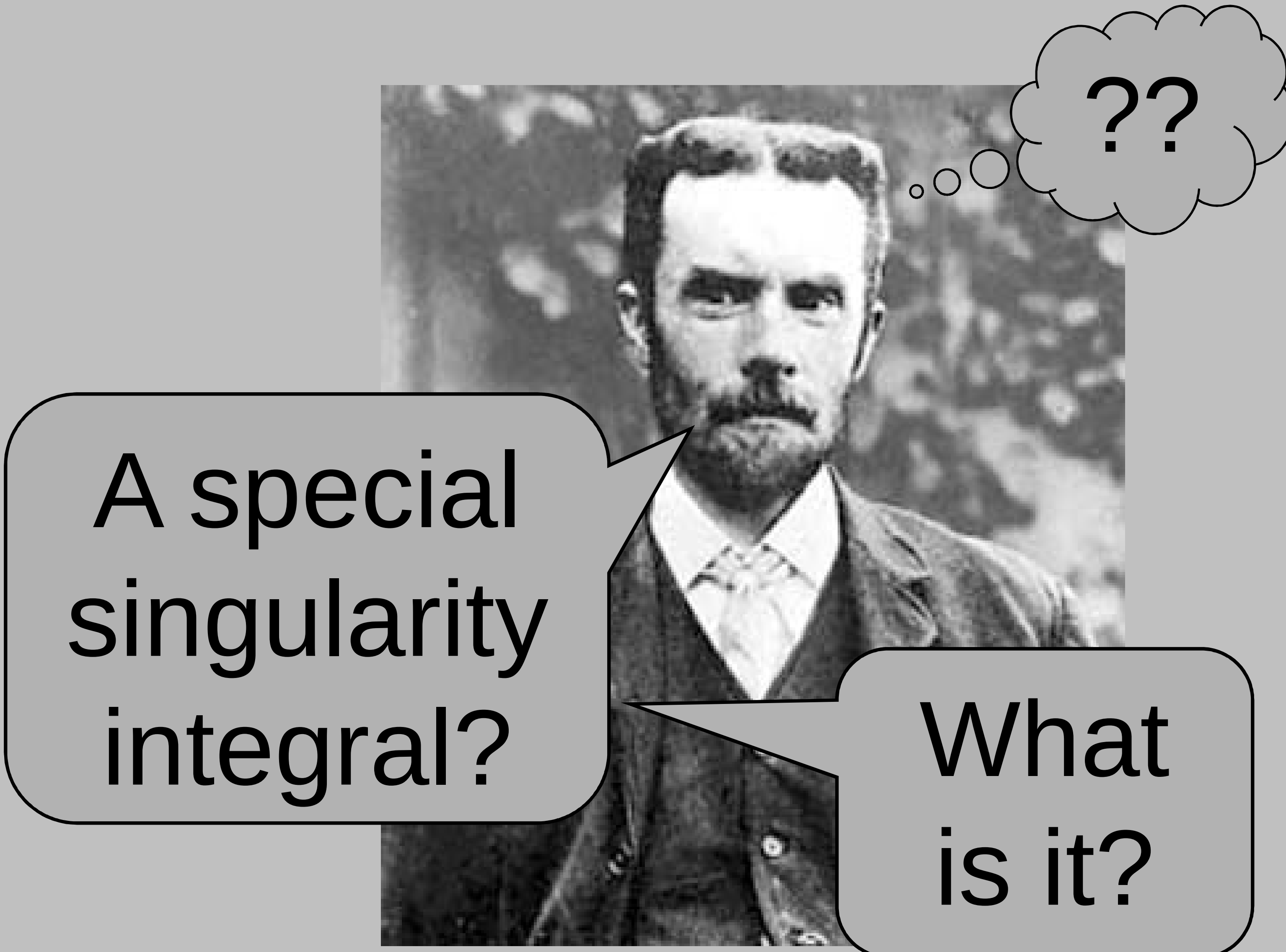
Impulse-Unit Step Relationship:

$$\delta(t) = \frac{du(t)}{dt}$$

or

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

Special Singularity Integral…



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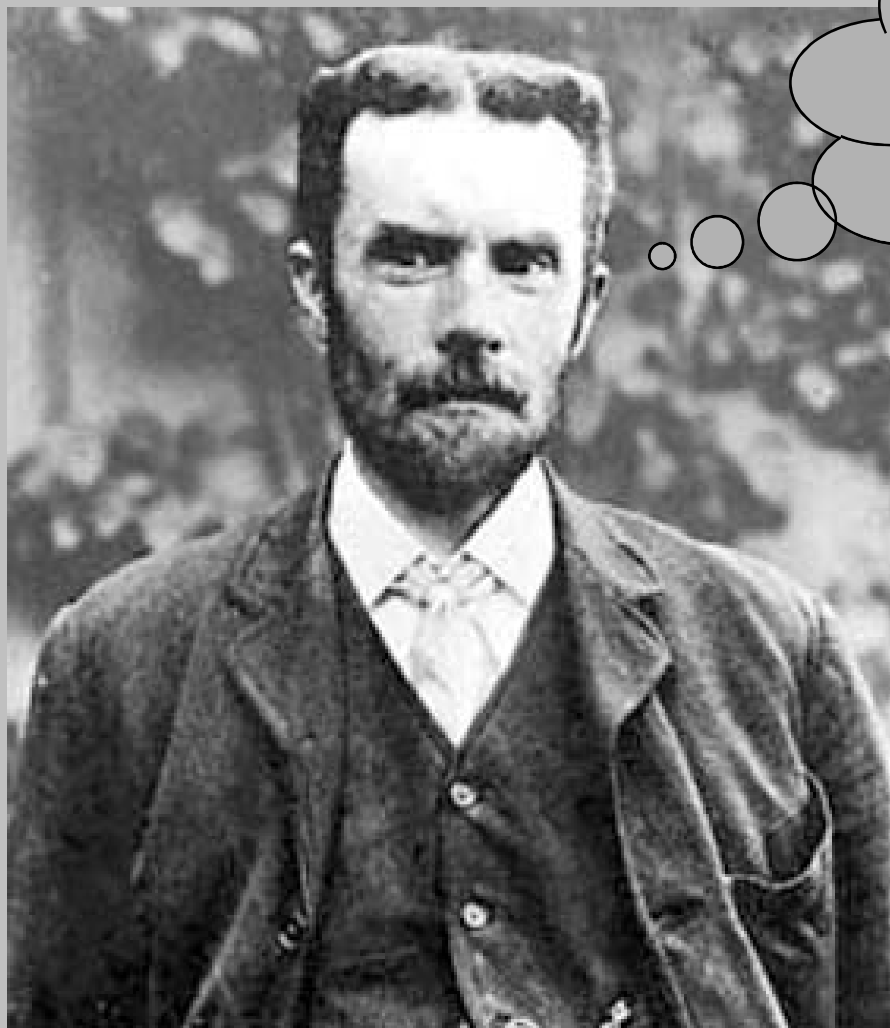
The Special Singularity Integral Involving $\delta(t)$ and $u(t)$ Product is:

$$\int_{0^-}^{0^+} u(t)\,\delta(t)\,dt = \frac{1}{2}$$



$$\int_{0^-}^{0^+} u(t)\,\delta(t)\,dt = \frac{1}{2}?$$

Simple Proof of the Special Singularity Integral:

$$\int_{0^-}^{0^+} u(t)\underbrace{\delta(t)dt}_{du(t)} = \int_{u(0^-)}^{u(0^+)} u(t)\,du(t)$$

$$= \frac{u^2(t)}{2}\Bigg|_{u(0^-)}^{u(0^+)} = \frac{1-0}{2} = \frac{1}{2}$$

Example: Calculation of Kinetic Energy Stored in a Rolling Body as a Result of an Impulsive Force

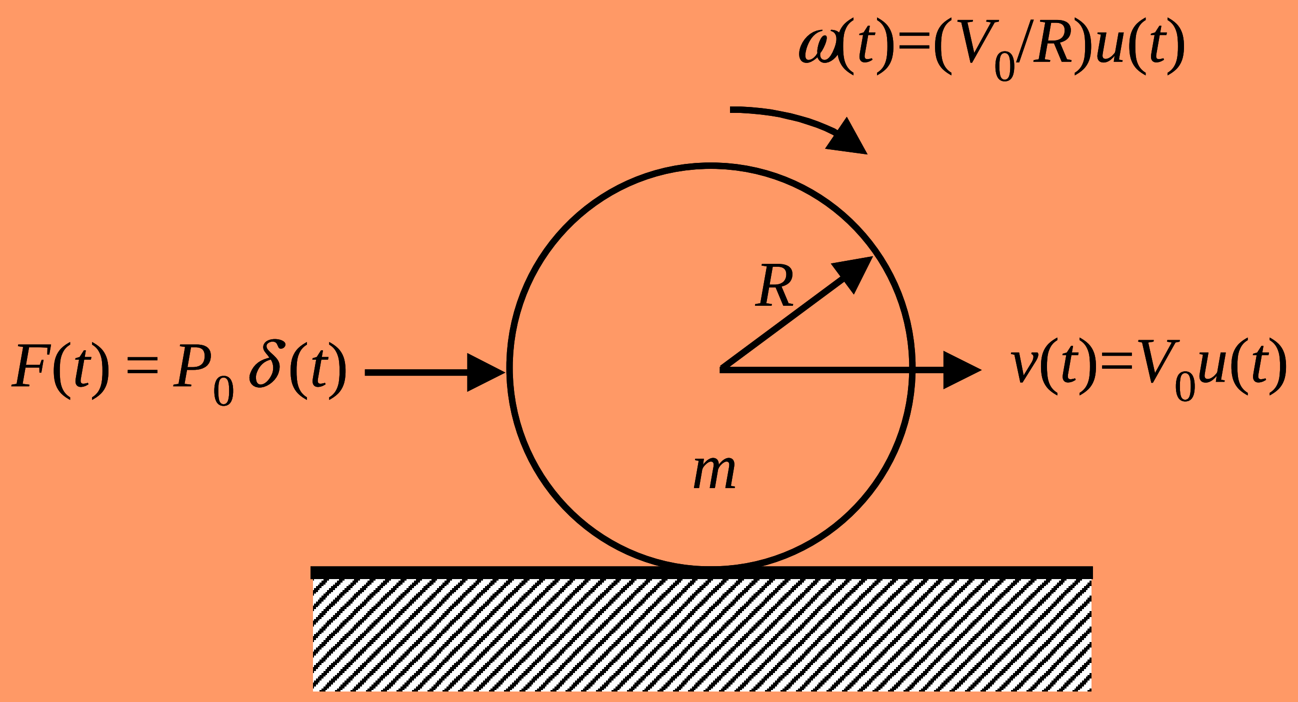


Fig 1. Force/body diagram.

Using Newton's Second Law of Motion to Obtain Translational Velocity Resulting from the Impulsive Force:

$$F(t) = m\frac{dv(t)}{dt} = P_0\delta(t)$$

$$v(t) = \frac{P_0}{m}\int_{0^-}^{t}\delta(t')\,dt' = \frac{P_0}{m}u(t) = V_0u(t)$$

Next, Obtain Rotational Angular Velocity and Torque Resulting from the Impulsive Force:

$$\omega(t) = \frac{v(t)}{R} = \frac{V_0}{R} u(t) = \omega_0 u(t)$$

$$\tau(t) = I_0 \frac{d\omega(t)}{dt} = I_0 \omega_0 \delta(t) = I_0 \left(\frac{V_0}{R}\right) \delta(t)$$

Total Kinetic Energy Delivered to the Object by the Impulsive Force:

$$W=\int_{0^-}^{\infty}\left[F(t)v(t)+\tau(t)\omega(t)\right]dt=\int_{0^-}^{0^+}\underbrace{P_0\,\delta(t)}_{F(t)}\underbrace{V_0u(t)}_{v(t)}dt$$

$$+\int_{0^-}^{0^+}\underbrace{I_0\,(V_0/R)\delta(t)}_{\tau(t)}\underbrace{(V_0/R)u(t)}_{\omega(t)}dt$$

$$=\left[P_0V_0+I_0(V_0/R)^2\right]\underbrace{\int_{0^-}^{0^+}u(t)\,\delta(t)\,dt}_{1/2}=\frac{1}{2}\underbrace{P_0}_{mV_0}V_0+\frac{1}{2}I_0\underbrace{(V_0/R)^2}_{\omega_0^2}$$

$$=\frac{1}{2}mV_0^2+\frac{1}{2}I_0\omega_0^2$$

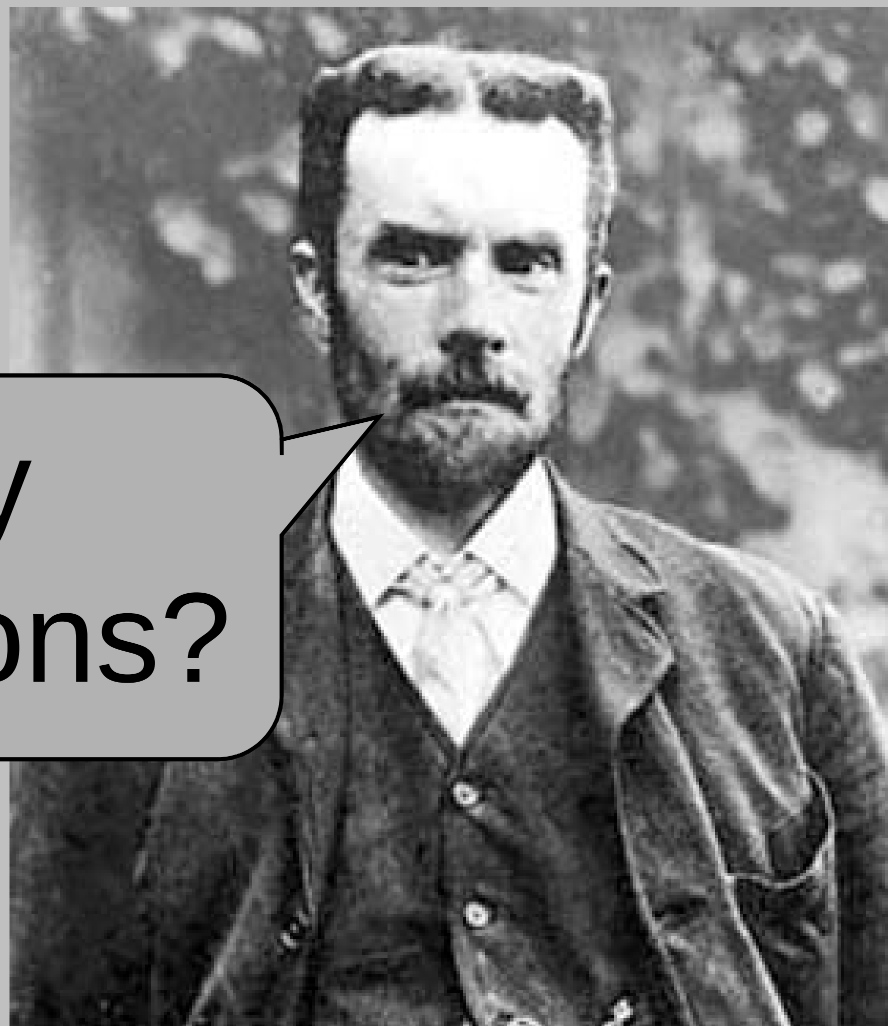
Conclusions:

- Extended simpler previous work to more general rolling body kinetic energy calculation (*both* translational plus rotational motion).

- The well-known calculation of kinetic energy stored in a rolling body due to a ***continuous*** force is extended to the case of ***discontinuous*** impulsive force using a special singularity integral.

- This ***discontinuous*** impulsive force case is done directly in a single, intuitive step. It makes clear where both the “1/2” factors come from, which is very instructive.

- This extended special case should be added to the literature.



Any questions?