Special Singularity Integrals Encountered in Electric Circuits

Aziz S. Inan & Peter M. Osterberg

School of Engineering

University of Portland

ISCAS 2005 (Kobe, Japan)

Impulse Function:

$$d(t) = 0 \quad \text{for} \quad t \neq 0$$

and

$$\int_{-\infty}^{\infty} d(t)\,dt = \int_{0^-}^{0^+} d(t)\,dt = 1.$$

$$d(0) \text{ is undefined.}$$

Unit Step Function:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0. \end{cases}$$

$u(0)$ is undefined.

Impulse-Step Relationship:

$$d(t) = \frac{du(t)}{dt}$$

or

$$u(t) = \int_{-\infty}^{t} d(t)\,dt$$

Doublet-Impulse Relationship:

$$d'(t) = \frac{dd(t)}{dt}$$

or

$$d(t) = \int_{-\infty}^{t} d'(t)\,dt\,.$$

Special Singularity Integral # 1 Involving $\delta(t)$ and $u(t)$ Product:

$$\int_{0^-}^{0^+} u(t)\underbrace{\delta(t)dt}_{du(t)} = \int_{u(0^-)}^{u(0^+)} u(t)\,du(t)$$

$$= \frac{u^2(t)}{2}\Bigg|_{u(0^-)}^{u(0^+)} = \frac{1-0}{2} = \frac{1}{2}.$$

Special Singularity Integral # 2 Involving $d(t)$ and $d'(t)$ Product:

$$\int_{0^-}^{0^+} d(t)\underbrace{d'(t)dt}_{dd(t)} = \int_{d(0^-)}^{d(0^+)} d(t)\, dd(t)$$

$$= \frac{d^2(t)}{2}\Bigg|_{d(0^-)}^{d(0^+)} = \frac{0-0}{2} = 0.$$

Special Singularity Integral # 3:

$$\int_{0^-}^{0^+} d^2(t)\,dt = \lim_{t \to 0} \int_{-t/2}^{t/2} \left(\frac{1}{t^2} \right) dt$$

$$= \lim_{t \to 0} \frac{1}{t} \to \infty.$$

Special Singularity Integral # 4:

$$\int_{0^-}^{0^+} u^2(t)\,dt = \lim_{t\to 0}\int_{-t/2}^{t/2}\left(\frac{t+t/2}{t}\right)^2 dt$$

$$= \lim_{t\to 0}\frac{(t+t/2)^3}{3t^2}\Bigg|_{-t/2}^{t/2}$$

$$= \lim_{t\to 0}\frac{t}{3} = 0.$$

Example # 1:

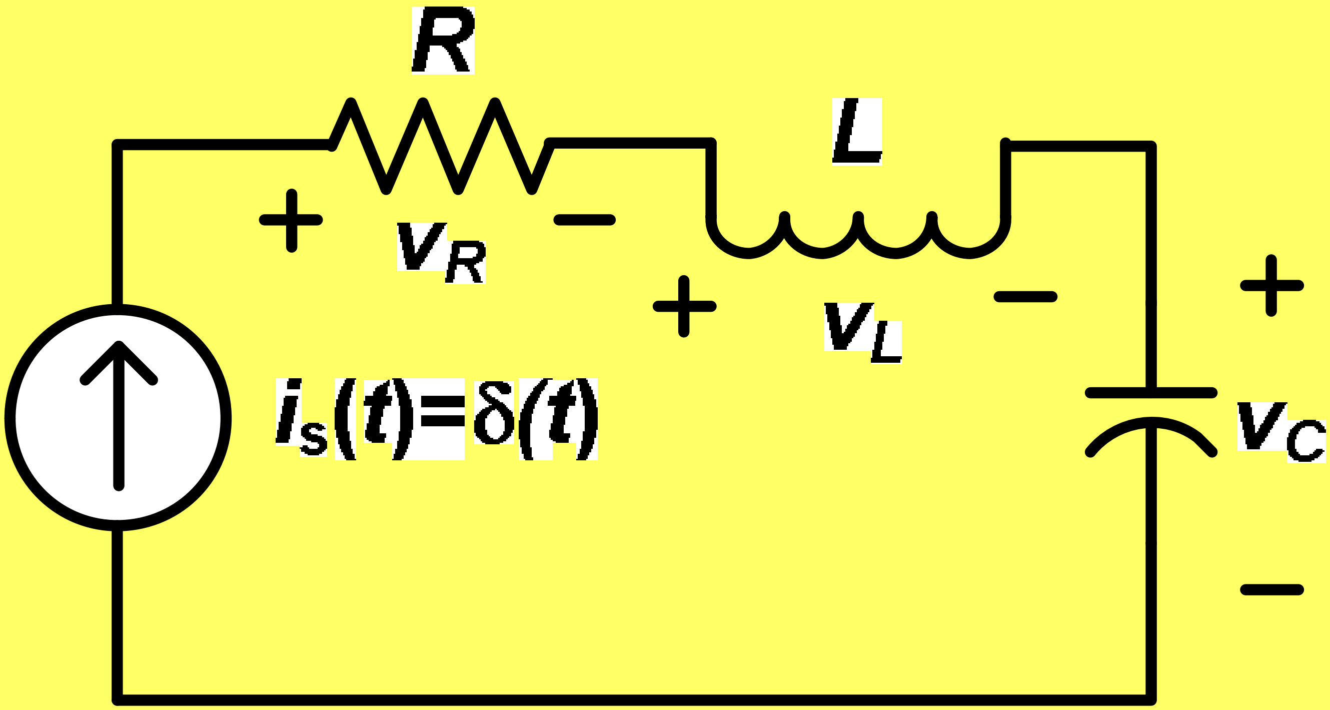

Fig. 1. Series *RLC* circuit excited by an impulse current source.

Voltage Across Each Element:

$$v_R(t) = Ri_S(t) = Rd(t)$$

$$v_L(t) = L\frac{di_S(t)}{dt} = L\frac{dd(t)}{dt} = Ld'(t)$$

$$v_C(t) = \frac{1}{C}\int_{0^-}^{t} i_S(t)\,dt = \frac{1}{C}\int_{0^-}^{t} d(t)\,dt = \frac{u(t)}{C}.$$

Energy of Each Element:

$$W_R = \int_{0^-}^{\infty} v_R(t) i_S(t)\,dt = R\int_{0^-}^{0^+} d^2(t)\,dt = \infty$$

$$W_L = \int_{0^-}^{\infty} v_L(t) i_S(t)\,dt = L\int_{0^-}^{0^+} d(t) d'(t)\,dt = 0$$

$$W_C = \int_{0^-}^{\infty} v_C(t) i_S(t)\,dt = \frac{1}{C}\int_{0^-}^{0^+} u(t) d(t)\,dt = \frac{1}{2C}.$$

Example # 2:

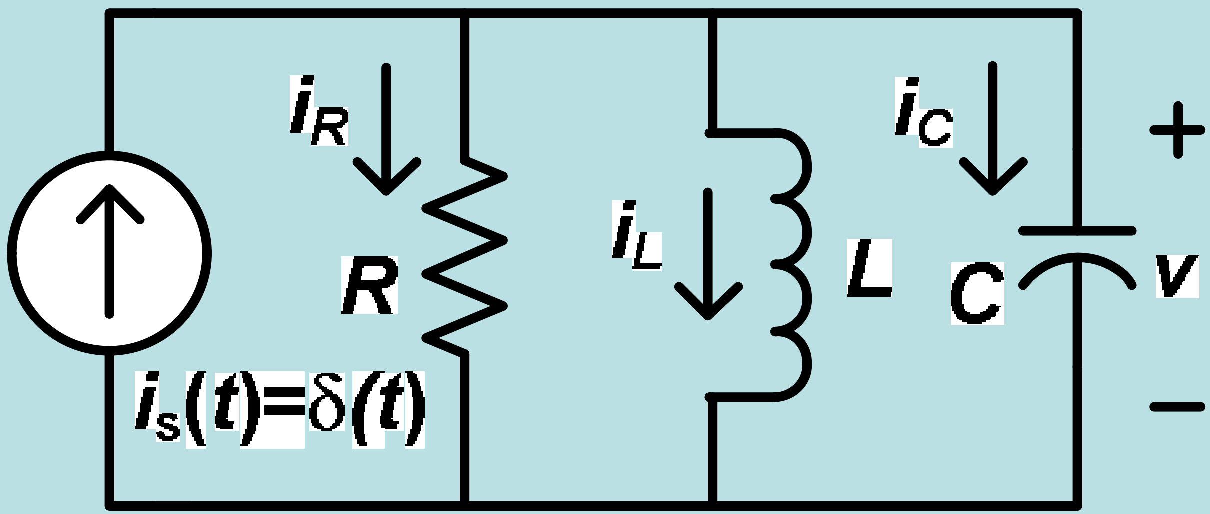

Fig. 2. Parallel *RLC* circuit excited by an impulse current source. Assume R=1/2 Ω, L=2/3 H, and C=1/2 F.

Governing Differential Equation:

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 3v = 2d'(t).$$

Voltage of Each Element:

$$v(t) = \left(3e^{-3t} - e^{-t}\right)u(t).$$

Current of Each Element:

$$i_R(t) = \frac{v(t)}{R} = \left(6e^{-3t} - 2e^{-t}\right)u(t)$$

$$i_L(t) = \frac{1}{L}\int_{0^-}^{t} v(t)\,dt = \left(\frac{3e^{-t}}{2} - \frac{3e^{-3t}}{2}\right)u(t)$$

$$i_C(t) = C\frac{dv(t)}{dt} = \left(-\frac{9e^{-3t}}{2} + \frac{e^{-t}}{2}\right)u(t) + d(t).$$

Total Energy Dissipated in the Resistor:

$$W_R = \int_{0^-}^{\infty} v(t) i_R(t) dt$$

$$= \underbrace{\int_{0^-}^{0^+} 8u^2(t) dt}_{0} + \underbrace{\int_{0^+}^{\infty} 2\left(3e^{-3t} - e^{-t}\right)^2 dt}_{1} = 1.$$

Total Energy Stored in the Inductor:

$$W_L = \int_{0^-}^{\infty} v(t) i_L(t) dt$$

$$= \underbrace{\int_{0^-}^{0^+} v(t) i_L(t) dt}_{0}$$

$$+ \underbrace{\int_{0^+}^{\infty} \left(6e^{-4t} - \frac{3}{2} e^{-2t} - \frac{9}{2} e^{-6t} \right) dt}_{0} = 0.$$

Total Energy Stored in the Capacitor:

$$W_C = \int_{0^-}^{\infty} v(t) i_C(t)\, dt$$

$$= \underbrace{\int_{0^-}^{0^+} 2\left[u(t)d(t) - 4u^2(t)\right] dt}_{1}$$

$$+ \underbrace{\int_{0^+}^{\infty} \left(6e^{-4t} - \frac{1}{2}e^{-2t} - \frac{27}{2}e^{-6t} \right) dt}_{-1} = 0.$$

Conclusions:

- Four special singularity integrals and their proofs are provided;
- Applications of these integrals are demonstrated in two circuit examples;
- Coverage of these integrals in the educational literature is recommended.

Any questions?