Special Singularity Integrals Encountered In Electric Circuits

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Abstract-**In this article, the authors first introduce four special singularity integral identities that involve the impulse, the doublet and the unit step functions, and provide a simple proof for each integral identity. Next, to demonstrate the use of these singularity integral identities, the authors consider the impulse response of two second-order *RLC* circuits and use these integrals to calculate the energy stored and dissipated in each circuit. The authors believe that these solutions are direct and easy to use in electrical circuit problems involving singularity functions. They hope that a wider coverage of these special integral identities and their applications will be offered in educational literature.**

I. Introduction

Singularity functions are fundamental to circuit theory. The three well-known singularity functions that will be considered in this article are the unit step function, the impulse function and the doublet. The impulse function constitutes the central member of singularity functions and is defined as

$$\delta(t)=0 \quad \text{for} \quad t\neq 0 \quad \text{and} \quad \int_{-\infty}^{\infty}\delta(t)dt=\int_{0^-}^{0^+}\delta(t)dt=1. \tag{1}$$

Note that $\delta(0)$ value is undefined. The impulse function is related to the unit step function $u(t)$ [1] as

$$\delta(t)=\frac{du(t)}{dt} \quad \text{or} \quad u(t)=\int_{-\infty}^{t}\delta(\tau)d\tau \tag{2}$$

where the unit step function is defined as

$$u(t)=\begin{cases}0, & t<0\\ 1, & t>0.\end{cases} \tag{3}$$

Note that $u(0)$ value is also undefined. The doublet $\delta'(t)$ is symbolically defined as the first derivative of the impulse function. The relationship between the impulse and the doublet is

$$\delta'(t)=\frac{d\delta(t)}{dt} \quad \text{or} \quad \delta(t)=\int_{-\infty}^{t}\delta'(\tau)d\tau. \tag{4}$$

Recently, the authors discovered four special singularity integral identities, each of which involves the product of two related singularity functions. In the next section, we will provide these special integral identities and their simple proofs.

II. Special Singularity Integrals

The first special singularity integral identity and its simple proof [2-5] are given by:

$$\int_{0^-}^{0^+}u(t)\underbrace{\delta(t)dt}_{du(t)}=\int_{u(0^-)}^{u(0^+)}u(t)du(t) = \left.\frac{u^2(t)}{2}\right|_{u(0^-)}^{u(0^+)}=\frac{1-0}{2}=\frac{1}{2}. \tag{5}$$

Note that the product $u(t)\delta(t)$ in (5), by itself, is meaningless, however, its integral given by (5) is valid, meaningful and applicable. Note also that (5) is correct regardless of the undefined value of the unit step function at its discontinuity point (i.e., $u(0)$).

The second special singularity integral identity and its simple proof based on (1) and (4) are as follows:

$$\int_{0^-}^{0^+}\delta(t)\underbrace{\delta'(t)dt}_{d\delta(t)}=\int_{\delta(0^-)}^{\delta(0^+)}\delta(t)\,d\delta(t) = \left.\frac{\delta^2(t)}{2}\right|_{\delta(0^-)}^{\delta(0^+)}=\frac{0-0}{2}=0. \tag{6}$$

Note that the product $\delta(t)\delta'(t)$ in (6), by itself, is meaningless, however, its integral given by (6) is valid, meaningful and applicable. Note also that (6) is valid regardless of the undefined value of $\delta(t)$ at $t=0$.

The third special singularity integral identity and its simple proof using unit-area rectangular sequence pulses of height $1/\tau$ and width τ over the range $t=-\tau/2$ to $t=\tau/2$ [1] are given by:

$$\int_{0^-}^{0^+}\delta^2(t)dt=\lim_{\tau\to 0}\int_{-\tau/2}^{\tau/2}\left(\frac{1}{\tau^2}\right)dt=\lim_{\tau\to 0}\frac{1}{\tau}\to\infty. \tag{7}$$

Note that the square of the impulse function $\delta^2(t)$, by itself, is meaningless. However, its integral given by (7) is valid, meaningful and applicable.

The fourth special integral identity and its simple proof where we interpret the unit step function in terms of ramp-step sequence functions that change from 0 to 1 with a slope of $1/\tau$ over the range $t=-\tau/2$ to $t=\tau/2$ [1] are as follows:

$$\begin{aligned}\int_{0^-}^{0^+}u^2(t)dt&=\lim_{\tau\to 0}\int_{-\tau/2}^{\tau/2}\left(\frac{t+\tau/2}{\tau}\right)^2dt\\&=\lim_{\tau\to 0}\left.\frac{(t+\tau/2)^3}{3\tau^2}\right|_{-\tau/2}^{\tau/2}=\lim_{\tau\to 0}\frac{\tau}{3}=0.\end{aligned} \tag{8}$$

Similar to $\delta^2(t)$, note that $u^2(t)$ is, by itself, meaningless. However, its integral given by (8) is valid, meaningful and applicable.

Next, the authors will consider two electrical circuit examples each excited by an impulsive current source to demonstrate the applications of the four special singularity integral identities (5) to (8).

III. Examples

A. Example 1-Impulse Response of a Series RLC Circuit

Consider the second-order series *RLC* circuit shown in Fig. 1 which is excited by an impulse current source given by $i_S(t)=\delta(t)$. We will assume the capacitor to be initially uncharged. Our goal is to determine the resulting energy dissipated or stored in each circuit element. Using the voltage and current relationships of basic passive elements, the voltage of each element can be found as follows:

$$\begin{aligned}v_R(t)&=Ri_S(t)=R\delta(t)\\v_L(t)&=L\frac{di_S(t)}{dt}=L\frac{d\delta(t)}{dt}=L\delta'(t)\\v_C(t)&=\frac{1}{C}\int_{0^-}^{t}i_S(\tau)d\tau=\frac{1}{C}\int_{0^-}^{t}\delta(\tau)d\tau=\frac{u(t)}{C}.\end{aligned} \tag{9}$$

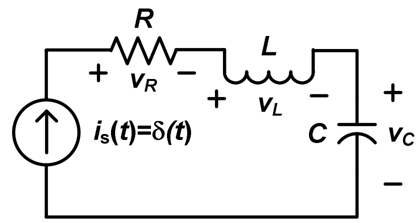

Fig. 1. Series *RLC* circuit excited by an impulse current source.

Note that the impulse current source resulted in an impulse voltage across the resistor, a doublet voltage across the inductor, and a step voltage across the capacitor. Using these voltages, the total energy dissipated or stored in each element in Fig. 1 due to the impulse current source is obtained directly using the special singularity integral identities (5) to (7) as

$$\begin{aligned}W_R&=\int_{0^-}^{\infty}v_R(t)i_S(t)dt=R\int_{0^-}^{0^+}\delta^2(t)dt=\infty\\W_L&=\int_{0^-}^{\infty}v_L(t)i_S(t)dt=L\int_{0^-}^{0^+}\delta(t)\delta'(t)dt=0\\W_C&=\int_{0^-}^{\infty}v_C(t)i_S(t)dt=\frac{1}{C}\int_{0^-}^{0^+}u(t)\delta(t)dt=\frac{1}{2C}.\end{aligned} \tag{10}$$

These results indicate that the impulse current that flows through the circuit shown in Fig. 1 causes infinite energy dissipation in the resistor, zero energy storage in the inductor, and a finite amount of energy storage equal to $1/(2C)$ in the capacitor, respectively.

B. Example 2-Impulse Response of a Parallel RLC Circuit

Next, we consider the second-order parallel *RLC* circuit excited by an impulse current source as shown in Fig. 2. Again, our goal is to determine the resulting energy dissipated or stored in each circuit element. Based on Kirchhoff's current law, the governing differential equation of the circuit expressed in terms of the voltage is given by:

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=\frac{1}{C}\delta'(t). \tag{11}$$

Before we provide the solution for (11), note that at $t=0$, since the inductor acts like an open circuit and the capacitor acts like a short circuit, the impulse source current flows entirely through the capacitor and causes the voltage of the circuit to jump instantaneously. So, we know intuitively that the voltage expression will change like a step function at $t=0$. The jump in the voltage is also expected due to the doublet that appears on the right side of (11) (i.e., the second derivative of the voltage on the left hand side of (11) must match this doublet on the right hand side). Next, selecting the values of the elements to be $R=1/2\ \Omega$, $L=2/3$ H, and $C=1/2$ F which result in an over-damped response with characteristic roots -3 and -1, the initial voltage due to the impulse current flow through the capacitor can be obtained as

$$v\left(0^{+}\right)=\frac{1}{C}\int_{0^{-}}^{0^{+}} i_C(t)dt=2\int_{0^{-}}^{0^{+}}\delta(t)dt=2. \tag{12}$$

Using the initial condition given by (12) along with the other initial condition which is $i_L(0^{+})=0$ (since the inductor initially acts like an open circuit), the solution of (11) is found to be

$$v(t)=\left(3e^{-3t}-e^{-t}\right)u(t). \tag{13}$$

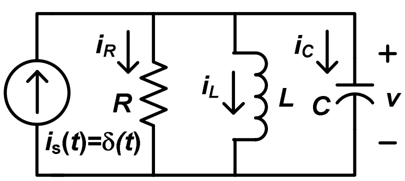

Fig. 2. Parallel *RLC* circuit excited by an impulse current source.

Using (13) along with the basic voltage-current relationships of passive elements, the current through each circuit element can be obtained as follows:

$$\begin{aligned} i_R(t)&=\frac{v(t)}{R}=\left(6e^{-3t}-2e^{-t}\right)u(t)\\ i_L(t)&=\frac{1}{L}\int_{0^{-}}^{t} v(\tau)d\tau=\left(\frac{3e^{-t}}{2}-\frac{3e^{-3t}}{2}\right)u(t)\\ i_C(t)&=C\frac{dv(t)}{dt}=\left(-\frac{9e^{-3t}}{2}+\frac{e^{-t}}{2}\right)u(t)+\delta(t). \end{aligned} \tag{14}$$

Note that over the time interval $t=0^{-}$ and $t=0^{+}$, (13) and (14) can be expressed as

$$\begin{aligned} v(t)&=2u(t)\\ i_R(t)&=4u(t)\\ i_L(t)&=0\\ i_C(t)&=-4u(t)+\delta(t). \end{aligned} \tag{15}$$

Therefore by using (5), (8), and (13) to (15), we can calculate the total energy dissipated or stored in each circuit element. The total energy dissipated in the resistor can be calculated as

$$\begin{aligned} W_R&=\int_{0^{-}}^{\infty} v(t)i_R(t)dt\\ &=\underbrace{\int_{0^{-}}^{0^{+}} 8u^2(t)dt}_{0}+\underbrace{\int_{0^{+}}^{\infty} 2\left(3e^{-3t}-e^{-t}\right)^2 dt}_{1}=1. \end{aligned} \tag{16}$$

The total final energy stored in the inductor can be found as

$$\begin{aligned} W_L&=\int_{0^{-}}^{\infty} v(t)i_L(t)dt\\ &=\underbrace{\int_{0^{-}}^{0^{+}} v(t)i_L(t)dt}_{0}\\ &\quad+\underbrace{\int_{0^{+}}^{\infty}\left(6e^{-4t}-\frac{3}{2}e^{-2t}-\frac{9}{2}e^{-6t}\right)dt}_{0}=0. \end{aligned} \tag{17}$$

The total final energy stored in the capacitor can be obtained as

$$\begin{aligned} W_C&=\int_{0^{-}}^{\infty} v(t)i_C(t)dt\\ &=\underbrace{\int_{0^{-}}^{0^{+}} 2\left[u(t)\delta(t)-4u^2(t)\right]dt}_{1}\\ &\quad+\underbrace{\int_{0^{+}}^{\infty}\left(6e^{-4t}-\frac{1}{2}e^{-2t}-\frac{27}{2}e^{-6t}\right)dt}_{-1}=0. \end{aligned} \tag{18}$$

Note that the portion of each integral from $t=0^{-}$ to $t=0^{+}$ given in (16) to (18) corresponds to either the initial energy dissipated or stored in each element. These energy values indicate that the impulse current source in the circuit shown in Fig. 2 result in a finite amount of initial energy storage in the capacitor while no initial energy is dissipated in the resistor nor stored in the inductor. As expected, the initial energy stored in the capacitor completely dissipates in the resistor as time approaches to infinity.

IV. CONCLUSIONS

In this paper, the authors first provide four special singularity integral identities including their proofs given by (5) to (8). Next, to demonstrate the use of these integral identities, the authors consider the impulse response of two second-order electric circuit examples in which they directly apply these integral identities to calculate the energy stored and dissipated in various elements of the circuit. The authors firmly believe that these special singularity integral identities are correct, easy to use and applicable to problems involving electric circuits as well as problems encountered in some other areas of engineering and physics [6]. Neither these integral identities nor their applications are currently covered in the educational literature (e.g., [7-18]). The authors hope that this article will serve as a vehicle to bring these special singularity integrals to the attention of the engineering community and to promote their wider use in the educational literature.

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