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REVISITING THE CALCULUS OF SINGULARITY FUNCTIONS

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ABSTRACT

Since the publication of Oliver Heaviside's work on symbolic calculus in 1893, scientists and engineers have made use of *singularity functions* and applied them to many different practical problems. Recently, the authors observe that some of the current literature involving singularity functions is inconsistent and/or incomplete and therefore raises some concern. In this article, the authors revisit the calculus of singularity functions in order to highlight these inconsistencies in an attempt to start a dialogue and ultimately unify the scientific community on this matter. Additionally, the proper solution of many interesting current and future practical scientific and engineering problems require the resolution of these inconsistencies. For these purposes, the authors consider two well-known singularity functions (the *unit impulse function* and the *Heaviside unit step function*), and review some of their properties and a number of *intimate* mathematical relationships and equations involving these two functions. The authors suggest that the current educational literature consider offering a wider coverage of these mathematical relations and equations.

INTRODUCTION

In physics, an *impulsive* quantity is generally considered to be a quantity of relatively large intensity or amplitude that is distributed over a relatively short range of some other quantity such as space or time. In mathematics, an *impulse function* is defined as a function that is an infinitely brief (or concentrated) and an infinitely strong pulse-shaped function having a finite area. Physically, while such a true impulsive quantity does not exist in the real world, the concept has been around for over a century in mathematical circles (Kirchhoff, 1882; Heaviside, 1893; Dirac, 1926; Dirac, 1958; Van der Pol and Bremmer, 1955; Lighthill, 1958; Papoulis, 1962; Bracewell, 2000[a]; Zemanian, 1965[a]; Siebert, 1986[a]; Jammer, 1989). The impulse function was first used by Gustav Kirchhoff (Kirchhoff, 1882). It was later defined and heavily applied in electromagnetic theory by Oliver Heaviside (Heaviside, 1893) and introduced into the early development of quantum mechanics by P. A. M. Dirac (Dirac, 1926; Dirac, 1958). The impulse function is an extremely powerful mathematical tool used to study the behavior of many phenomena involving impulsive quantities. For example, a brief force in physics is often represented in mathematical discussion by an impulse function of time.

The *unit impulse function* $\delta(x)$ (also known as a *delta* or *Dirac delta* function) is mathematically defined as:

$$\delta(x)=0 \text{ for } x\neq 0 \text{ and } \int_{-\infty}^{\infty}\delta(x)\,dx=1 \qquad (1)$$

Note however that $\delta(x)$ is not a function in the ordinary sense since no value is supplied for it at $x=0$. The most important equation involving $\delta(x)$ is the well-known *sifting* integral given by:

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0) \qquad (2)$$

where $f(x)$ is assumed to be a continuous function of x at $x=0$.

The unit impulse function constitutes the central member of a special family of functions of infinite members that are *intimately* related one to the other through differentiation and integration. This family of functions is collectively known as *singularity functions* 'since, in the light of conservative mathematics, their behavior is rather singular' (Guillemin, 1953). Any one of these functions is denoted by the symbol $u_n(x)$ where the subscript n is an integer and is referred to as the order of the singularity function (Guillemin, 1953; Lynch and Truxal, 1962[a]; Huang and Parker, 1971; Director, 1975; Scott, 1965; Ziemer et al., 1998; Oppenheim et al., 1997). Using this notation, for example, the unit impulse function $\delta(x)$ is designated as $u_0(x)$, or as the singularity function of order zero. Some might view singularity functions as a very special group of functions and, therefore, can only be applied to a restricted class of *theoretical* problems. However, contrary to this view, the authors believe that singularity functions can be used in a much wider range of *practical* applications due to the fact that any arbitrary function can be expressed in terms of appropriately chosen singularity functions.

Recently, while the authors have been working on new applications of singularity functions in physics and engineering (Osterberg and Inan, 1999; Osterberg and Inan, 2000; Inan and Osterberg, 2000[a]; Inan and Osterberg, 2001), interestingly enough, they observed that some of the past and current literature on the applications and calculus of singularity functions is somewhat scattered, contradictory and raises some concern (Siebert, 1986[b]; Mita and Boufaida, 1999; Gangopadhyaya and Mallow, 2000; Vibet, 1999; Paskusz, 2000; Inan and Osterberg, 2000[b]; Griffiths and Walborn, 1999). This fact motivated the authors to revisit and investigate the calculus of singularity functions, pose some fundamental questions, provide or suggest some answers to these questions, and make recommendations on some common set of rules and guidelines on this matter. The authors by no means claim that the content of this paper is final but rather view it as an initial step in opening up a dialog among scholars who have expertise in this field. The authors wish that this article plays a crucial role to invite and motivate the experts to participate in the conversation on the calculus of singularity functions and hope that the scientific community will reunite and agree toward accepting a common set of rules and guidelines on this matter.

THE CALCULUS OF SINGULARITY FUNCTIONS –REVISITED

In this section, the authors revisit the calculus of singularity functions by posing three fundamental questions based on their observations on the use of singularity functions in the literature. The first two questions are basic and non-controversial, whose answers are well established among the scientific community. However, the third question is more difficult and controversial and there are differences of opinion among the scientific community in providing concrete answers for these questions. The authors provide some answers for some of these questions and suggest or advise possible answers for the others. The following is the list of the three questions posed by the authors followed by their responses on each.

Question 1: ***What basic singularity functions exist and are accepted by the scientific community?***

In general, there are an infinite number of *basic* singularity functions that exist and are accepted which are represented by $u_n(x)$ where the subscript n is referred to as the *order* of the singularity function and can take any integer value in the range $-\infty \leq n \leq \infty$. Successive basic singularity functions are defined by (Guillemin, 1953; Lynch and Truxal, 1962[a]; Huang and Parker, 1971; Director, 1975; Scott, 1965; Ziemer et al., 1998):

$$u_{n+1}(x) = \frac{du_n(x)}{dx} \tag{3}$$

The unit impulse function is the central singularity function of order zero designated as $u_0(x)$ (also commonly designated as $\delta(x)$ as stated previously). Another well-known member of the singularity function family is the *Heaviside unit step function* $u_{-1}(x)$ (also commonly designated as $u(x)$) defined as:

$$u_{-1}(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \tag{4}$$

Note that, generally, the value of the unit step function at the jump point $x = 0$ (i.e., $u_{-1}(0)$) is undefined (Van der Pol and Bremmer, 1955; Director, 1975; Scott, 1965; Ziemer et al., 1998; Oppenheim et al., 1997; Bracewell, 2000[b]). Some other singularity functions are the *unit parabolic function* $u_{-3}(x)$, *unit ramp function* $u_{-2}(x)$ (also commonly designated as $r(x)$), the *unit doublet* $u_1(x)$ (also commonly designated as $\delta'(x)$) and the *unit triplet* $u_2(x)$ (also designated as $\delta''(x)$). Note that for all basic singularity functions, it is assumed that $u_n(x) = 0$ for $x < 0$. Also note that $u_n(x)$ for $n \geq -2$ constitutes the special group of singularity functions that are not ordinarily differentiable everywhere.

Question 2: ***What mathematical operations involving basic singularity functions are valid and accepted by the scientific community?***

Note that for this question, the authors refer to simple functions which are linear combinations of individual *basic singularity functions* of the form:

$$f(x) = \sum_{n=-\infty}^{\infty} b_n u_n(x) \tag{5}$$

where $u_n(x)$ represents the different types of basic singularity functions and b_n represents the constant coefficients.

2.1. *Differentiation* of basic singularity functions is allowed as stated previously where successive basic singularity functions are related by differentiation as given by Eq. (3). As an example, one can provide the well-known *intimate* relationship between $u_{-1}(x)$ and $u_0(x)$ as (Van der Pol and Bremmer, 1955; Bracewell, 2000[a]; Siebert, 1986[a]; Temple, 1981):

$$\frac{du_{-1}(x)}{dx} = u_0(x) \tag{6}$$

or, equivalently:

$$\frac{du(x)}{dx} = \delta(x) \tag{7}$$

Eq. (7) states that the derivative of the unit step function is the unit impulse function. Since the unit step function has a jump and does not in fact possess a derivative at the origin, Eq. (7) must be interpreted as shorthand for 'the derivatives of a sequence of differentiable functions that approach $u(x)$ as a limit and constitute a suitable defining sequence for $\delta(x)$' (Bracewell, 2000[a]). Note that Eq. (7) can easily be extended to the derivative of a simple discontinuous function $f(x)$ which can be written in terms of the unit step function $u(x)$ as:

$$f(x) = g(x) + ku(x) = \begin{cases} g(x) & x < 0 \\ g(x) + k & x > 0 \end{cases} \tag{8}$$

where $g(x)$ is assumed to be an ordinary continuous function and $f(x)$ is undefined at $x = 0$ since $u(0)$ is undefined. The derivative of $f(x)$ results in a term $k\,\delta(x)$ and is given by:

$$\frac{df(x)}{dx} = \frac{dg(x)}{dx} + k\,\delta(x) \tag{9}$$

Eq. (9) can easily be extended to the derivatives of simple functions with multiple unit step discontinuities, consisting of summation of terms as $k_i u(x - x_i)$, where the derivative of each discontinuity located at each point x_i with an amount of jump k_i leads to a term $k_i \delta(x - x_i)$.

2.2. *Integration* of basic singularity functions is also allowed. Similar to the case of differentiation, successive basic singularity functions are related to one another through integration as (Lynch and Truxal, 1962[a]; Huang and Parker, 1971; Director, 1975; Scott, 1965; Ziemer et al., 1998):

$$\int_{-\infty}^{x} u_n(x')dx' = u_{n-1}(x) \tag{10}$$

For example, the *intimate* relationship between the unit impulse function and the unit step function which simply results from their definitions given by Eqs. (1) and (4) can also be written as (Van der Pol and Bremmer, 1955; Bracewell, 2000[a]):

$$\int_{-\infty}^{x} u_0(x')dx' = u_{-1}(x) \tag{11}$$

or, equivalently:

$$\int_{-\infty}^{x} \delta(x')dx' = u(x) \tag{12}$$

Based on Eq. (12), the definite integral of the unit impulse function is the unit step function. Eq. (12) can easily be verified by interpreting $\delta(x)$ as the limit of an appropriate sequence of functions (Bracewell, 2000[a]). It is interesting to note here that based on Eqs. (3) and (10), successive differentiation of basic singularity functions can be undone through successive integration (Guillemin, 1953). Furthermore, differentiation of any simple function of the form given by Eq. (5) followed by an integration based on Eq. (10) or vice versa (i.e., integration followed by differentiation) will recover the original simple function.

2.3. Singularity functions can be used as input signals in *linear constant-coefficient ordinary differential equations* which have the general form as (Lynch and Truxal, 1962[b]; Zemanian, 1965[b]):

$$\sum_{k=0}^{N} a_k \frac{d^k y(x)}{dx^k} = \sum_{n=-\infty}^{\infty} b_n u_n(x) \tag{13}$$

Eq. (13) is an ordinary differential equation with order N where the right-hand-side represents the superposition of basic singularity input signals $u_n(x)$ and $y(x)$ is the output signal which is produced as a result of those inputs. Eq. (13) is typically encountered in physical problems involving linear time-invariant systems. For example, the governing differential equation of a first-order *RC* circuit excited by a unit step voltage signal starting at $t = 0$ can be written as:

$$\frac{dy(t)}{dt} + \frac{1}{\tau}y(t) = \frac{1}{\tau}u(t) \tag{14}$$

where the output signal $y(t)$ represents the capacitor voltage and τ is the time constant of the circuit given by $\tau = RC$. The solution of this first-order differential equation under zero initial condition is referred to as the unit step response of the circuit given by the well-known expression:

$$y(t) = \left(1 - e^{-t/\tau}\right)u(t) \tag{15}$$

Similarly, the unit impulse response of this *RC* circuit can be obtained by replacing the right-hand-side of Eq. (14) with $(1/\tau)\,\delta(t)$ and re-solving to obtain:

$$y(t) = (1/\tau)e^{-t/\tau}u(t) \tag{16}$$

2.4. *Sampling property of an impulse function*. It can easily be shown by considering sequences of pulses that the product of the function $f(x)$ with the unit impulse function for the case when $f(x)$ is continuous at $x = 0$ yields (Dirac, 1958; Bracewell, 2000[a]; Siebert, 1986[a]):

$$f(x)\delta(x) = f(0)\delta(x) \tag{17}$$

Eq. (17) is called the *sampling* property of the unit impulse function. Therefore, the sampling property of an impulse function $\delta(x)$ is valid when $f(x)$ is continuous at $x = 0$.

2.5. *Sifting property of an impulse function*. It can easily be shown using Eq. (17) that the integral of the product of the function $f(x)$ with the unit impulse function $\delta(x)$ for the case when $f(x)$ is continuous at $x = 0$ yields (Dirac, 1958; Van der Pol and Bremmer, 1955; Lighthill, 1958; Papoulis, 1962; Bracewell, 2000[a]; Zemanian, 1965[a]; Siebert, 1986[a]):

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0) \qquad (18)$$

Eq. (18) is called the *sifting* property of the unit impulse function. Therefore, the sifting property of an impulse function $\delta(x)$ is valid when $f(x)$ is continuous at $x = 0$.

Question 3: ***What functions, properties, and mathematical operations of singularity functions are considered to be questionable and controversial?***

Note that in this question, the authors consider more complex singularity functions along with their properties and operations where the discussion is kept limited to expressions involving only the *unit step* and *unit impulse* functions.

3.1: Regarding the *existence and validity of* $u(0)$, the authors believe that, in general, the value of the unit step function at the jump point does not exist and should be left undefined (Van der Pol and Bremmer, 1955; Director, 1975; Scott, 1965; Ziemer et al., 1998; Oppenheim et al., 1997; Bracewell, 2000[b]). This is consistent with the observation that the unit ramp function $r(x)$, whose derivative is the unit step function $u(x)$, is not differentiable at its corner point $x = 0$ (i.e., the slope at $r(0)$ does not have a single definite value) (Thomas, 2001). However, it is also interesting to note that in the special case when one decides to choose an *even* sequence of pulse functions to represent the unit impulse function in the limit, then, based on Eq. (12), the integral of the even pulse sequence in the limit will lead to a corresponding unit step function which will have a definite value at $x = 0$, given by $u(0) = 1/2$.

3.2. *Singularity functions consisting of terms such as* $u^p(x)$, $e^{u(x)}$ *and* $sin[u(x)]$ are valid functions. As an example, consider the discontinuous function $f(x) = e^{u(x)}$ given by:

$$f(x) = e^{u(x)} = \begin{cases} 1 & x < 0 \\ e & x > 0 \end{cases} \qquad (19)$$

and undefined at $x = 0$, since $u(0)$ is undefined.

3.3. Regarding *the issue of equality of two similar singularity functions* such as $u^p(x)$ and $u^q(x)$ (where it is assumed that $p \geq 1$ and $q \geq 1$), consider as an example the two discontinuous functions $u(x)$ and $u^2(x)$ (Vibet, 1999; Paskusz, 2000; Inan and Osterberg, 2000[b]; Craig, 1964). These two functions are equal to zero for $x < 0$ and equal to one for $x > 0$. However, they are, in general, not equal at $x = 0$ since $u(0) \neq u^2(0)$ except in the special cases when $u(0)$ happens to be either one or zero. Therefore, the authors suggest that $u^p(x)$ and $u^q(x)$ be considered not equal.

3.4. *Sampling property of an impulse function when* $f(x)$ *is discontinuous at* $x = 0$ is invalid. The authors believe that unlike Eq. (17), in the case when $f(x)$ function is discontinuous and has a jump at $x = 0$, the product $f(x)\delta(x)$ *by itself* cannot be given a consistent meaning and must be avoided simply because $f(0)$ is undefined (Siebert, 1986[a]; Craig, 1964).

3.5. *Regarding the differentiation of discontinuous functions consisting of terms such as* $u^p(x)$, $e^{u(x)}$ *and* $sin[u(x)]$, consider as an example the discontinuous function $f(x) = u^p(x)$. Differentiating this function yields

$$\frac{df(x)}{dx} = pu^{p-1}(x)\delta(x) \qquad (20)$$

As before, the authors believe that the right-hand-side of Eq. (20) is, *by itself,* meaningless and must be avoided since $u^{p-1}(0)$ is undefined. Similarly, consider a second example where $f(x) = e^{u(x)}$. We differentiate $f(x)$ to obtain:

$$\frac{df(x)}{dx} = e^{u(x)}\delta(x) \qquad (21)$$

which again leads to an expression that is ambiguous since $e^{u(0)}$ is undefined. Therefore, the authors suggest that the differentiation of discontinuous functions consisting of terms such as $u^p(x)$, $e^{u(x)}$ and $\sin[u(x)]$ be considered invalid.

3.6. *Regarding the integration of discontinuous functions consisting of terms such as* $u^p(x)$, $e^{u(x)}$ *and* $sin[u(x)]$, consider as an example the discontinuous function $f(x) = u^p(x)$. Integrating $f(x)$, we get:

$$\int_{-\infty}^{x} u^p(x')dx' = r(x) \qquad (22)$$

Note that the result of the integration operation given by Eq. (22) is independent of the value of the function at the discontinuity point, $f(0)$, which is undefined. Therefore, unlike the differentiation operation, integration of more complicated discontinuous functions is valid. However, note that unlike in the case of basic singularity functions (i.e., when $p = 1$), differentiating the result of Eq. (22) in general *will not* recover the function $u^p(x)$ (i.e., $dr(x)/dx = u(x) \neq u^p(x)$).

3.7. *Sifting property of an impulse function when* $f(x)$ *is discontinuous at* $x = 0$ is valid. We already know that when the function $f(x)$ is continuous at $x = 0$, the well-known sifting integral given by Eq. (18) "sifts out" $f(0)$ value. Now, what happens to Eq. (18) if $f(x)$ is discontinuous and has a jump at $x = 0$? Under this circumstance, Eq. (18) is not valid since the function value $f(0)$ is undefined. However, the authors firmly believe that the integral portion (left-hand-side) of Eq. (18) taken by itself can still be carried out and takes a valid and meaningful limiting value. As an example, consider the discontinuous function given by $f(x) = u^p(x)$. Even if the function value $f(0)$ at the discontinuity point is undefined (since $u(0)$ is undefined), the value of the sifting integral of $f(x)$ can be obtained using Eq. (7) as:

$$\int_{-\infty}^{\infty} u^p(x)\,\delta(x)\,dx = \int_{-\infty}^{\infty} u^p(x)\,du(x) = \frac{1}{p+1}u^{p+1}(x)\Big|_{-\infty}^{\infty} = \frac{1}{p+1} \tag{23}$$

Furthermore, note that in the special case when $f(x)$ is a simple discontinuous function which can be expressed in the form as Eq. (8), the sifting integral takes on a special value which is the arithmetic average of the values of the discontinuous $f(x)$ function just before and just after $x = 0$, given by (Bracewell, 2000[a]):

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = [f(0^+)+f(0^-)]/2 \tag{24}$$

A recent article in the literature claims that Eq. (24) "cannot be sustained in general and continues to embarrass the unwary" (Griffiths and Walborn, 1999). The authors disagree with this confusing claim and firmly believe that Eq. (24) is always valid under the special condition that the discontinuous function $f(x)$ has the form given by Eq. (8). For example, consider a special case of Eq. (8) when $g(x) = 0$ and $k = 1$ (i.e., the discontinuous function is simply equal to the unit step function given by $f(x) = u(x)$), Eq. (24) then becomes:

$$\int_{-\infty}^{\infty} u(x)\,\delta(x)\,dx = 1/2 \tag{25}$$

The authors observe that this special integral equation given by Eq. (25) is completely missing in the educational literature such as in signals and systems and electrical circuits textbooks. Furthermore, there are numerous publications in the literature which claim that the above special singularity integral on the left-hand-side of Eq. (25) is ambiguous, meaningless, does not exist and/or must be avoided (Siebert, 1986[b]; Mita and Boufaida, 1999; Gangopadhyaya and Mallow, 2000). Some publications even make wrong and inconsistent assumptions to avoid this singularity integral (e.g., assume $u^2(x) = u(x)$) (Vibet, 1999; Paskusz, 2000) which is, in general, not correct (Inan and Osterberg, 2000[b]; Craig, 1964). Contrary to these claims, the authors firmly believe that the singularity integral equation given by Eq. (25) is *always* valid as long as $u(x)$ and $\delta(x)$ are related by Eqs. (7) and (12) and can also be verified by interpreting both $u(x)$ and $\delta(x)$ as a limit of appropriate sequence of functions (Inan and Osterberg, 2001). The authors also recently discovered that this special singularity integral equation is very useful and has direct applications to many special practical problems in various branches of science and engineering (Osterberg and Inan, 1999; Osterberg and Inan, 2000; Inan and Osterberg, 2000[a]; Inan and Osterberg, 2001). In addition, the limiting value of the general sifting integral (i.e., Eq. (24)) applied to a simple discontinuous function of the form given by Eq. (8) can be verified by using Eq. (18) and the special integral in Eq. (25) as:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\delta(x)dx &= \int_{-\infty}^{\infty}[g(x)+ku(x)]\delta(x)dx\\ &= g(0)+k/2\\ &= \underbrace{[g(0)+k]/2}_{f(0^+)/2}+\underbrace{g(0)/2}_{f(0^-)/2}\\ &= [f(0^+)+f(0^-)]/2\end{aligned} \tag{26}$$

CONCLUSION

In this article, the authors made an attempt to revisit the calculus of singularity functions, by surveying some of the past and current literature on the calculus and applications of singularity functions and by providing their views and observations on the mathematical inconsistencies on this matter. The authors by no means claim that their analysis of the mathematics and use of singularity functions is final and believe that more research is definitely needed to reunite the scholars and experts on this matter. The authors sincerely hope that this work will help the scientific community to gain a more uniform understanding and use of singularity functions in the literature.

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