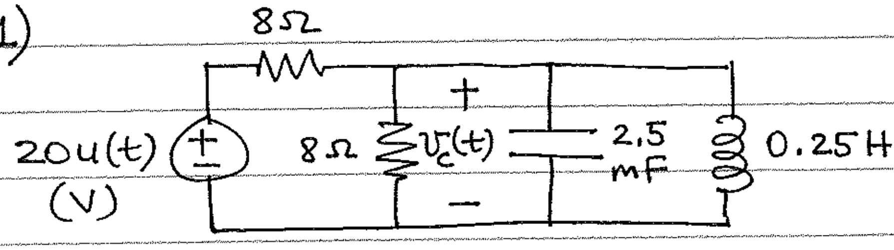
EE 261
FALL 2011
A. INAN

SECOND-ORDER RLC CIRCUITS

STEP EXCITATION PROBLEMS DONE DURING CLASS

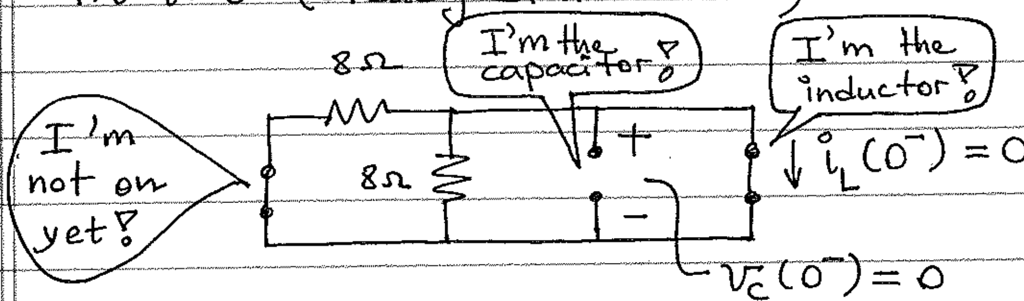
(#1)



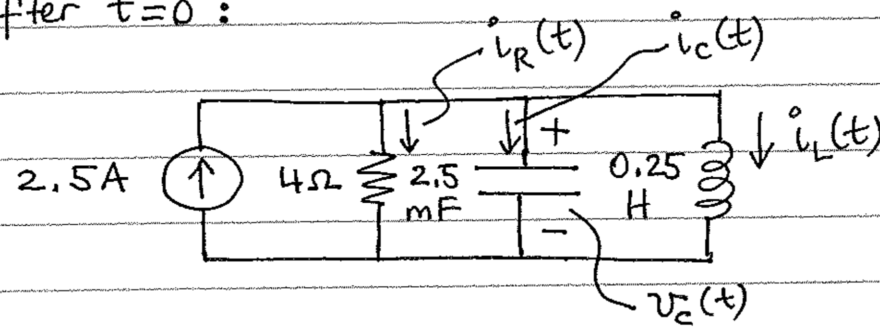
Find $v_c(t)$ for $t \geq 0$.

Solution:

At $t=0^-$ (steady state holds):



After $t=0$:



$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \rightarrow \underbrace{s^2 + 100s + 1600 = 0}_{(s+20)(s+80)}$$

$\therefore$ Overdamped response $\rightarrow v_c(t) = A_1 e^{-20t} + A_2 e^{-80t} + A_3$

$$v_c(0^+) = \boxed{A_1 + A_2 + A_3 = 0} \leftarrow (\mathrm{I})$$

Using KCL:

$$i_L(t) = 2.5 - i_C(t) - i_R(t)$$

$$= 2.5 - C\frac{dv_C(t)}{dt} - \frac{v_C(t)}{R}$$

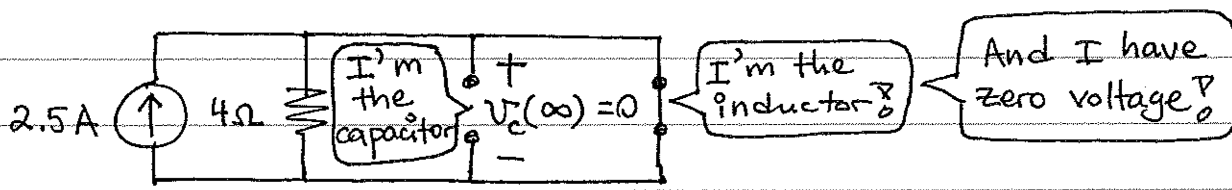
$$= 2.5 - 2.5\times10^{-3}\left[-20A_1e^{-20t} - 80A_2e^{-80t}\right] - \frac{\left[A_1e^{-20t} + A_2e^{-80t} + A_3\right]}{4}$$

$$\therefore\ i_L(0^+) = 2.5 + \cancel{0.05A_1} + \cancel{0.2A_2} - \cancel{0.25A_1} - \cancel{0.25A_2} - 0.25A_3$$

$$= 2.5 - 0.2A_1 - 0.05A_2 - 0.25A_3 = 0$$

or $\boxed{4A_1 + A_2 + 5A_3 = 50}$ $\leftarrow$ (II)

At $t\to\infty$ (steady state holds):

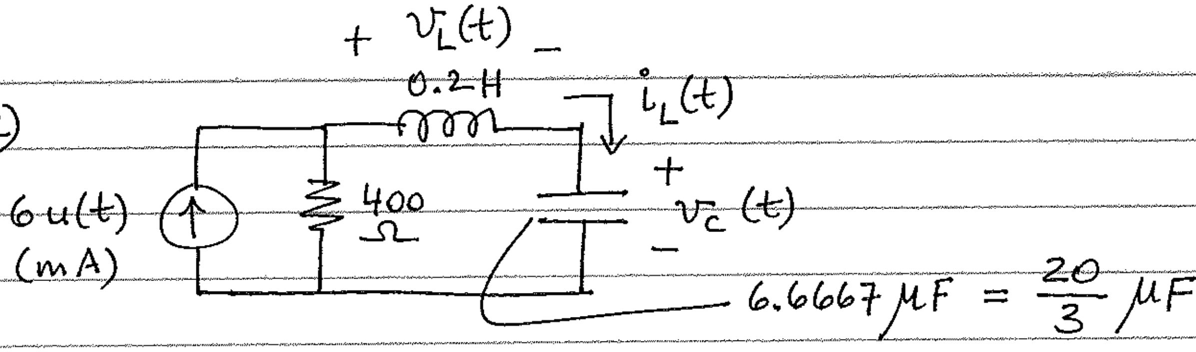


$v_C(t\to\infty) = \boxed{A_3 = 0}$ $\leftarrow$ (III)

Solving (I), (II) & (III) simultaneously yields:

$$\boxed{v_C(t) = \frac{50}{3}e^{-20t} - \frac{50}{3}e^{-80t}\ \text{(V)},\ \text{for } t \geq 0}$$

(#2)



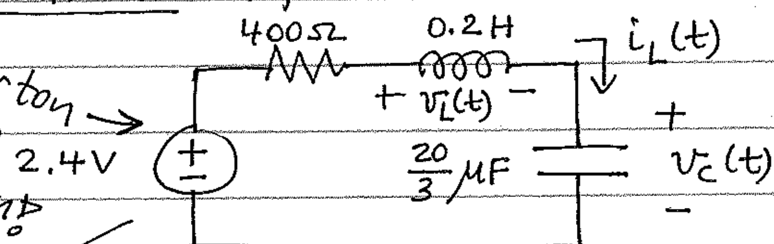
Given $v_C(0^-) = 2\text{V}$ & $i_L(0^-) = 8$ mA, find

(a) $v_C(t)$ for $t \geq 0$

(b) $v_L(t)$ for $t > 0$

Solution: After $t=0$:

Using Norton to Thevenin transformation



Second-order series RLC circuit, so using table:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \leftarrow \text{Characteristic equation}$$

$$s^2 + \frac{400}{0.2}s + \frac{1}{(0.2)(6.6667\mu)} = 0 \rightarrow s^2 + 2000s + 750{,}000 = 0$$

$$\rightarrow (s+500)(s+1500) = 0 \rightarrow s_1, s_2 = -500, -1500$$

∴ Overdamped response since real and different characteristic roots, therefore:

$$v_C(t) = A_1 e^{-500t} + A_2 e^{-1500t} + A_3$$

Using the first initial condition:

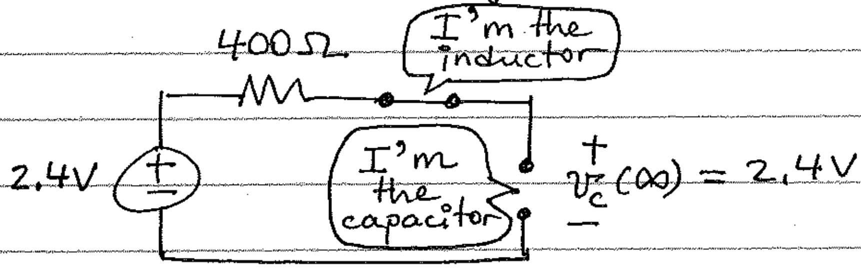
$$v_C(0^+) = \boxed{A_1 + A_2 + A_3 = 2} \leftarrow (\text{I})$$

$$i_L(t) = i_c(t) = C\,\frac{dv_c(t)}{dt} = \underbrace{\frac{20}{3}\times 10^{-6}}_{6.6667\,\mu F}\left[-500A_1e^{-500t} - 1500A_2e^{-1500t}\right]$$

$$i_L(0^+) = \frac{20}{3}\times 10^{-6}\left[-500A_1 - 1500A_2\right] = 8\times 10^{-3}$$

$$\longrightarrow \boxed{A_1 + 3A_2 = -12} \leftarrow (\text{II})$$

At $t \to \infty$ (steady state holds):



$$v_c(t\to\infty) = \boxed{A_3 = 2.4\text{ V}} \leftarrow (\text{III})$$

Then, solving (I), (II) and (III) simultaneously:

$$\left.\begin{aligned} A_1 + A_2 &= -0.4 \\ A_1 + 3A_2 &= -12 \end{aligned}\right\} \rightarrow A_1 = 5.4 \;\&\; A_2 = -5.8$$

$$\boxed{\therefore\; v_c(t) = 5.4e^{-500t} - 5.8e^{-1500t} + 2.4 \text{ (V), for } t \geq 0}$$

Next, to find $v_L(t) = L\,\dfrac{di_L(t)}{dt}$ where $i_L(t) = i_c(t)$, we first find $i_c(t) = C\,\dfrac{dv_c(t)}{dt}$:

$$i_c(t) = \frac{20}{3}\times 10^{-6}\left[-2700e^{-500t} + 8700e^{-1500t}\right]$$

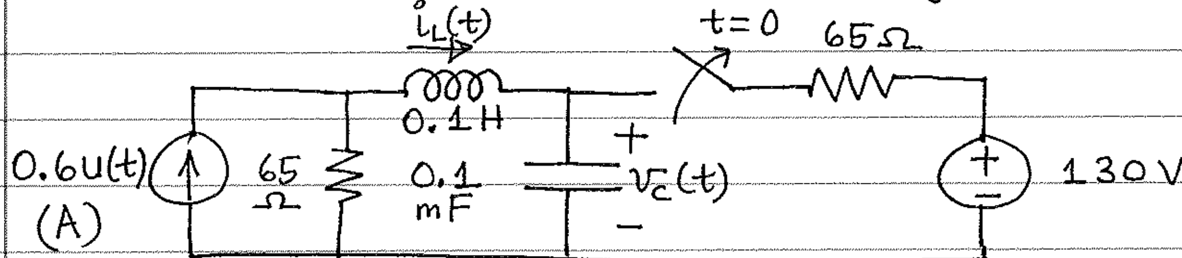
$$i_c(t) = -0.018\,e^{-500t} + 0.058\,e^{-1500t} \quad (A)$$

Next, we find $v_L(t) = L\,\dfrac{di_c(t)}{dt}$:

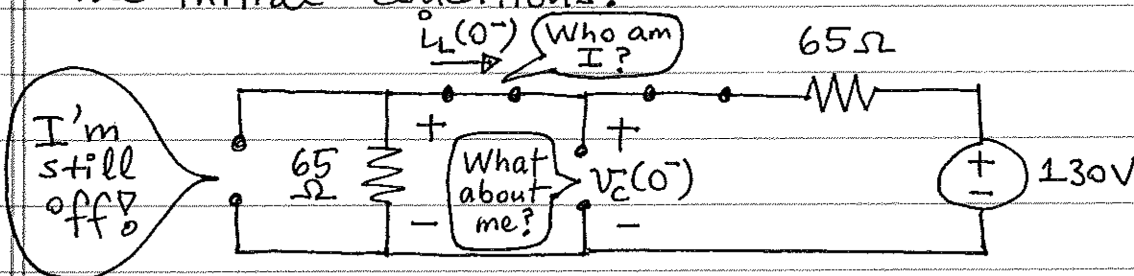
$$v_L(t) = 0.2\left[9e^{-500t} - 87e^{-1500t}\right]$$

$$\therefore \boxed{v_L(t) = 1.8\,e^{-500t} - 17.4\,e^{-1500t}, \quad \text{for } t > 0}$$

(#3) For the electric circuit shown, find $v_c(t)$ for $t \geq 0$. Assume steady state at $t = 0^-$.



Solution: Draw the circuit at $t = 0^-$ to obtain the initial conditions:

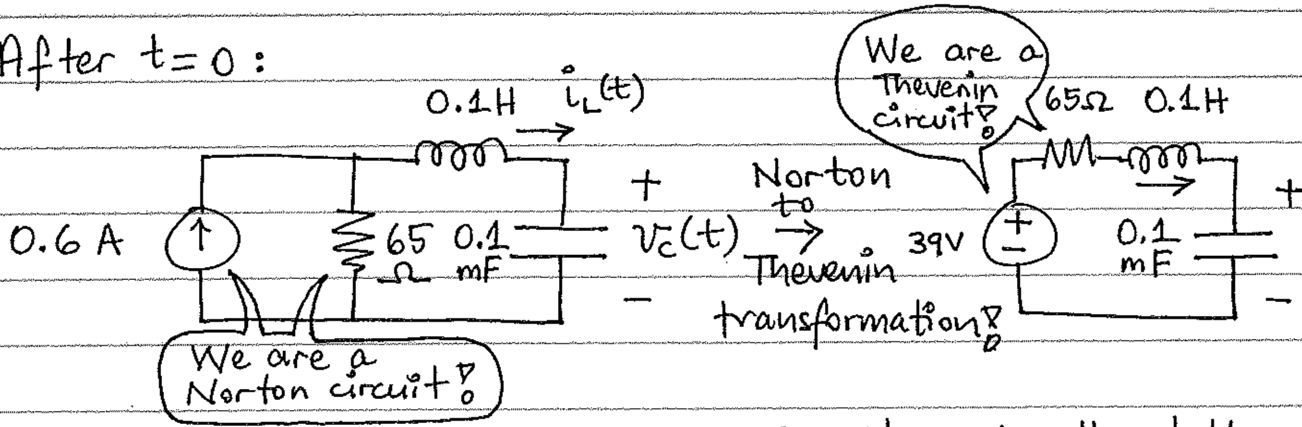


Using VDP → $v_c(0^-) = \dfrac{65}{65+65}(130) = 65\text{ V}$

$i_L(0^-) = -\dfrac{65\text{V}}{65\,\Omega} = -1\text{ A}$

initial conditions!

After $t=0$:



Since second-order series RLC circuit, using the table, characteristic equation can be obtained as

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \rightarrow s^2 + \frac{65}{0.1}s + \frac{1}{(0.1)(0.1m)} = 0$$

$$\rightarrow \underbrace{s^2 + 650s + 100,000}_{(s+250)(s+400)} = 0$$

$\therefore$ Overdamped response:

$$v_c(t) = A_1 e^{-250t} + A_2 e^{-400t} + A_3, \text{ for } t \geq 0$$

Using the initial conditions:

$$v_c(0^+) = \boxed{A_1 + A_2 + A_3 = 65} \leftarrow (\text{I})$$

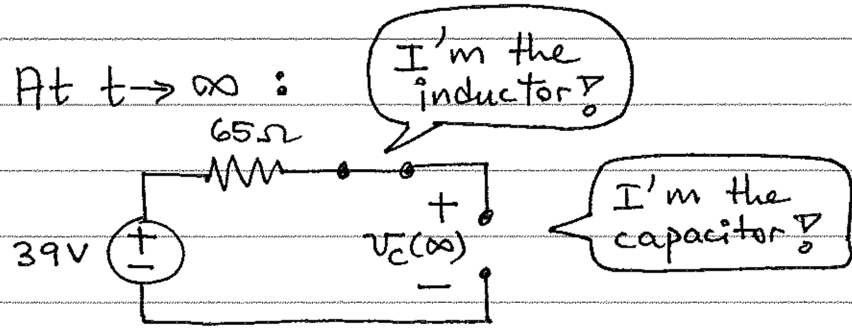
Since inductor and capacitor are connected in series:

$$i_L(t) = i_c(t) = C\frac{dv_c(t)}{dt}$$

$$= 10^{-4}\left[-250A_1 e^{-250t} - 400A_2 e^{-400t}\right]$$

$$i_L(0^+) = -0.025A_1 - 0.04A_2 = -1 \rightarrow \boxed{5A_1 + 8A_2 = 200} \quad (\text{II})$$

At $t \to \infty$:



$\rightarrow v_C(\infty) = 39\,V$

$$\therefore\ v_C(\infty) = \boxed{A_3 = 39\ V} \leftarrow (\mathrm{III})$$

Solving (I), (II) and (III) simultaneously yields:

$$\left.\begin{aligned} A_1 + A_2 &= 26 \\ 5A_1 + 8A_2 &= 200 \end{aligned}\right\} 5(26 - A_2) + 8A_2 = 200$$

$$\downarrow$$

$$3A_2 = 70 \rightarrow \boxed{A_2 = \frac{70}{3}}$$

$$\downarrow$$

$$\boxed{A_1 = \frac{8}{3}}$$

$$\boxed{\therefore\ v_C(t) = \frac{8}{3}e^{-250t} + \frac{70}{3}e^{-400t} + 39\ \ (V),\ \text{for } t \geq 0}$$