University ☺f P☺rtland
Sch☺☺l ☺f Engineering

EE 261
Fall 2011
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Solutions to Homework # 8–
Arithmetic of Complex Numbers and Phasor Analysis
(Friday, December 9, 2011)

P 1. Rectangular to polar form. Write the following rectangular-form complex numbers in polar form:

1.a. $V_1 = 5\sqrt{2} + j5\sqrt{2} = \underline{10e^{j45^\circ} \text{ or } 10e^{j(\pi/4)(\text{in radians})}}$

1.b. $Z_2 = 120j + 50 \cong \underline{130e^{j67.4^\circ} \text{ or } 130e^{j0.374\pi}}$

1.c. $Y_3 = j0.02 = \underline{0.02e^{j90^\circ} \text{ or } 0.02e^{j\pi/2}}$

1.d. $I_4 = -10\sqrt{3} - j10 = \underline{20e^{-j150^\circ} \text{ or } 20e^{-j5\pi/6}}$

1.e. $Z_5 = -j6{,}000 + 8{,}000 \cong \underline{10{,}000e^{-j36.87^\circ} \text{ or } 10{,}000e^{-j0.205\pi}}$

1.f. $V_6 = -2.5 = \underline{2.5e^{j(\pm 180^\circ)} \text{ or } 2.5e^{j(\pm\pi)}}$

P 2. Polar to rectangular form. Convert the following polar-form complex numbers in rectangular form:

2.a. $I_1 = 10e^{j180^\circ} = -10 + j0 = \underline{-10}$

2.b. $V_2 = 3\sqrt{2}e^{-j45^\circ} = \underline{3 - j3}$

2.c. $Z_3 = 200e^{-j90^\circ} = 0 - j200 = \underline{-j200}$

2.d. $Y_4 = \left(e^{-j2\pi/3}\right)/500 = (1/500)\left(-1/2 - j\sqrt{3}/2\right) = \underline{-0.001 - j0.001\sqrt{3}}$

2.e. $V_5 = 2.3e^{j5\pi/6} = 2.3\left(-\sqrt{3}/2 + j/2\right) = \underline{-1.15\sqrt{3} + j1.15}$

2.f. $I_6 = 5.6\times10^{-3}e^{j240^\circ} = 5.6\times10^{-3}\left(-1/2 - j\sqrt{3}/2\right) = \underline{-2.8\times10^{-3} - j2.8\times10^{-3}\sqrt{3}}$

P 3. Phasor representation of sinusoidal signals. Find the phasor-form representation of the following sinusoidal signals. (Note that the capital letters represent the phasor form.)

3.a. $i_1(t) = 2\times10^{-4}\cos\left(2\pi\times10^4 t + 60^\circ\right) \xrightarrow{\text{phasor rep.}} \underline{I_1 = 2\times10^{-4} e^{j60^\circ}}$

3.b. $v_2(t) = 3.8\cos\left(10^6 t - 25^\circ\right) \rightarrow V_2 = \underline{3.8e^{-j25^\circ}}$

3.c. $v_a(t) = 1.73\sin\left(10^5 \pi t + \pi/10\right) \rightarrow V_a = 1.73e^{j(\pi/10 - \pi/2)} = \underline{1.73e^{-j2\pi/5}}$

3.d. $i_c(t) = -4.2\sin\left(3\pi\times10^5 t - 3\pi/5\right) \rightarrow I_c = 4.2e^{j(\pi/2 - 3\pi/5)} = \underline{4.2e^{-j\pi/10}}$

3.e. $i_s(t) = 0.05\cos\left(6.28\times10^4 t - 135^\circ\right) \rightarrow I_s = \underline{0.05e^{-j135^\circ}}$

3.f. $v_s(t) = 10\sin\left(8\pi\times10^4 t - 300^\circ\right) \rightarrow V_s = 10e^{j(-300^\circ - 90^\circ)} = \underline{10e^{-j30^\circ}}$

3.g. $i_T(t) = 0.03\cos\left(10^5 t - 135^\circ\right) - 0.015\sin\left(10^5 t + 60^\circ\right)$

$\rightarrow I_T = \underline{0.03e^{-j135^\circ} - 0.015e^{-j30^\circ}}$

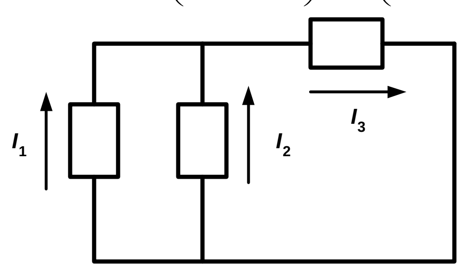
3.h. $v_T(t) = 2.1\sin\left(4.4\pi\times10^4 t - 30^\circ\right) + 4.9\cos\left(4.4\pi\times10^4 t + 30^\circ\right)$

$\rightarrow V_T = \underline{2.1e^{-j120^\circ} + 4.9e^{j30^\circ}}$

P 4. Basic arithmetic operations of complex numbers. Solve the following circuit problems, simplify each answer and provide it in polar form.

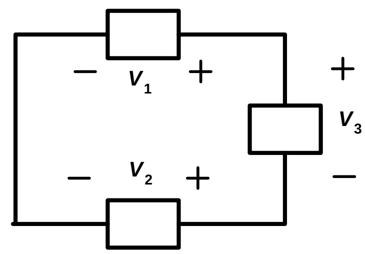
4.a. Kirchhoff's current law (KCL) applied in the phasor domain. Given $I_1 = 10e^{j30^\circ}$, $I_2 = 10e^{-j30^\circ}$, $I_3 = I_1 + I_2 = ?$ (Note that each I_k current is a phasor quantity which represents a real-time **S**inusoidal **S**teady-**S**tate (SSS) current $i_k(t)$ flowing in the circuit shown below. Just like the time-domain currents, the phasor-domain currents must also satisfy KCL.)

__**Solution:**__ Transforming polar-form phasor currents into rectangular form, we can find the phasor current I_3 as $I_3 = I_1 + I_2 = 10\left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right) + 10\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) = \underline{10\sqrt{3} + j0 = 10\sqrt{3}e^{j0}}$.



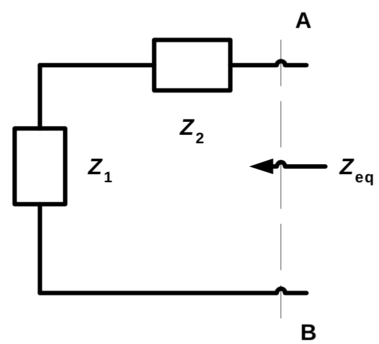
4.b. Kirchhoff's voltage law (KVL) in phasor form. In the phasor-domain circuit shown below, $V_1 = 5\sqrt{2}e^{-j45^\circ}$, $V_2 = 5\sqrt{2}e^{j45^\circ}$, $V_3 = V_1 - V_2 = ?$ (Note that each V_k is a phasor quantity which represents a real-time SSS voltage $v_k(t)$ in the circuit shown below. The phasor-domain voltages must satisfy KVL.)

Solution: Using rectangular-form phasors V_1 and V_2, the phasor voltage V_3 can be found as $V_3 = V_1 - V_2 = 5\sqrt{2}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right) - 5\sqrt{2}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) = \underline{0 - j10 = 10e^{-j90^\circ}}$.



4.c. Equivalent impedance. The circuit shown between terminals A and B is shown in phasor domain. If $Z_1 = (200 - 400j)\,\Omega$ and $Z_2 = (200 + j100)\,\Omega$, $Z_3 = Z_1 + Z_2 = ?$ (Note that each Z represent an impedance. Impedances can be combined in series or in parallel just like resistances.)

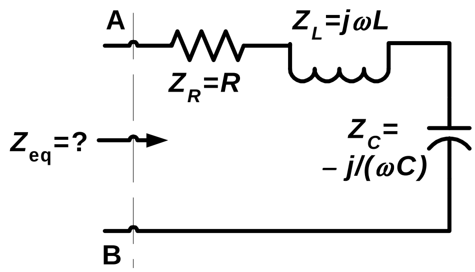
Solution: Since the two impedances are connected in series, the equivalent impedance Z_{eq} is given by $Z_{eq} = Z_1 + Z_2 = (200 - 400j) + (200 + j100) = \underline{400 - j300 \cong 500e^{-j36.87^\circ}\ \Omega}$.



4.d. Equivalent impedance of a series *RLC* circuit. In the series *RLC* circuit shown, the element values are given by $R = 4\,\Omega$, $L = 5\,\text{mH}$, and $C = 1.25\,\text{mF}$ respectively. Find the equivalent impedance of this circuit at three different frequencies: $\omega_1 = 200\,\text{rad/s}$, $\omega_2 = 400\,\text{rad/s}$, and $\omega_3 = 1{,}200\,\text{rad/s}$.

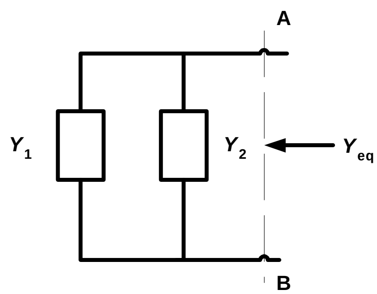
Solution: Since the three impedances are connected in series, the equivalent impedance Z_{eq} at any frequency is given by $Z_{eq}(\omega) = Z_R + Z_L(\omega) + Z_C(\omega) = 4 + j\left(5\omega \times 10^{-3} - \frac{800}{\omega}\right)$.

Substituting the frequency values $\omega_1 = 200\,\text{rad/s}$, $\omega_2 = 400\,\text{rad/s}$, and $\omega_3 = 1{,}200\,\text{rad/s}$, we find $Z_{eq}(\omega_1) = \underline{(4 - j3)\,\Omega}$, $Z_{eq}(\omega_2) = \underline{4\,\Omega}$, and $Z_{eq}(\omega_3) \cong \underline{(4 + j5.33)\,\Omega}$ respectively.



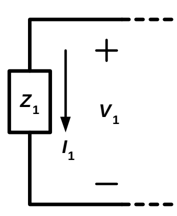
4.e. Equivalent admittance. Admittance of an element represented by Y (in Siemens) is defined as the inverse of the impedance Z (in Ω) of the same element, i.e., $Y = Z^{-1}$. Given $Y_1 = 0.002e^{j\pi/2}$ S, $Y_2 = 0.002\sqrt{2}e^{-j\pi/4}$ S, what is $Y_3 = Y_1 + Y_2 = ?$ (Note that admittances can be combined in series or in parallel just like the same way as conductances.)

<u>Solution:</u> Since the two admittances are connected in parallel, the equivalent admittance Y_{eq} is given by $Y_{\text{eq}} = Y_1 + Y_2 = \underbrace{0 + j0.002}_{Y_1} + \underbrace{0.002 - j0.002}_{Y_2} = \underline{0.002 \cong 0.002e^{j0}\text{ S}}$.



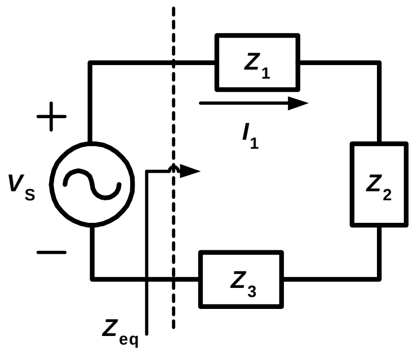
4.f. Ohm's law in phasor form. Given $I_1 = 0.02e^{-j\pi/3}$ A, $Z_1 = 150e^{j\pi/6}$ Ω, $V_1 = Z_1 I_1 = ?$ (Note that $V = ZI$ is the phasor-domain equivalent of the time-domain Ohm's law given as $v_R(t) = Ri_R(t)$. Note also that Ohm's law in phasor form is not only limited to resistors but can also be used for inductors and capacitors.)

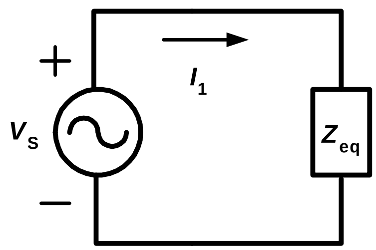
<u>Solution:</u> Using Ohm's law in phasor form, the phasor voltage V_1 can be obtained as $V_1 = \left(150e^{j\pi/6}\right)\left(0.02e^{-j\pi/3}\right) = \underline{3e^{-j\pi/6}\text{ V}}$.



4.g. Phasor-domain solution of sinusoidal steady-state circuits. In the circuit shown, given $V_S = 4e^{j30^\circ}$ V, $I_1 = 0.02e^{-j23.13^\circ}$ A, $Z_1 = (40 + j80)\,\Omega$, $Z_2 = (30 - j20)\,\Omega$, $Z_3 = ?$

Solution: Using Ohm's law in phasor form, we can find the equivalent impedance Z_{eq} seen from the source as $Z_{eq} = V_S / I_1 = 4e^{j30^\circ} / 0.02e^{-j23.13^\circ} = 200e^{j53.13^\circ} = 120 + j160\,\Omega$. However, since the three impedances are connected in series, Z_{eq} is also given by $Z_{eq} = Z_1 + Z_2 + Z_3 = (40 + j80) + (30 - j20) + Z_3 = 120 + j160\,\Omega$ from which we can obtain the unknown impedance Z_3 as $\underline{Z_3 = (50 + j100)\,\Omega}$.





4.h. Thevenin impedance. For the phasor-domain circuit shown, given the three impedance values to be $Z_1 = 50\,\Omega$, $Z_2 = j50\,\Omega$, $Z_3 = -j50\,\Omega$, $Z_{Th} = Z_3 + \dfrac{Z_1 Z_2}{Z_1 + Z_2} = ?$

Solution: The Thevenin impedance seen between terminals A and B can be calculated as

$$Z_{Th} = -j50 + \frac{(50)(j50)}{50 + j50} = -j50 + \frac{j50}{1 + j} = -j50 + \frac{j50(1 - j)}{\underbrace{(1 + j)(1 - j)}_{1+1}} = -j50 + 25 + j25 = \underline{25 - j25\,\Omega}$$

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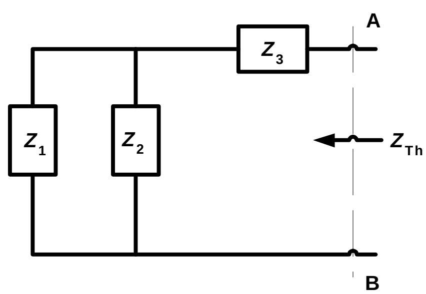
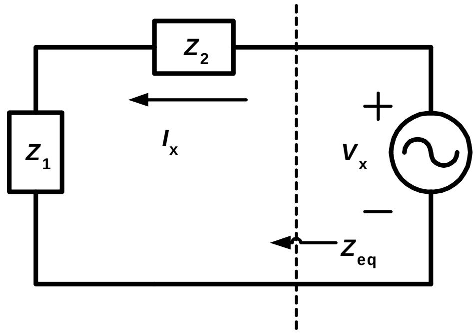


4.i. KVL and Ohm's law. In the phasor-domain circuit shown below, given $Z_1 = (300 + j300)\,\Omega$, $Z_2 = 300\sqrt{2}e^{-j\pi/4}\,\Omega$, $I_x = 0.04e^{j\pi/3}$ A, $V_x = (Z_1 + Z_2)I_x = ?$

Solution: The phasor-form source voltage V_x can be calculated using Ohm's law and the equivalent impedance seen between the terminals of the voltage source as

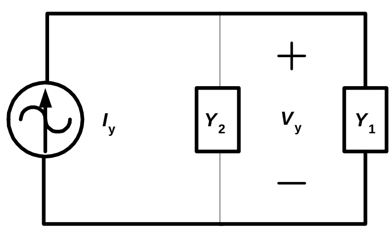
$$V_x = Z_{eq} I_x = \left(\underbrace{300 + j300}_{Z_1} + \underbrace{300 - j300}_{Z_2}\right)\underbrace{\left(0.04e^{j\pi/3}\right)}_{I_x} = (600\,\Omega)\left(0.04e^{j\pi/3}\text{ A}\right) = \underline{24e^{j\pi/3}\text{ V}}.$$



4.j. KCL and Ohm's law (optional). In the following phasor-domain circuit shown, $Y_1 = 0.02e^{-j\pi/2}$ S, $Y_2 = 0.01\sqrt{3}e^{j\pi/3}$ S, $V_y = e^{j60^\circ}$ V, $I_y = (Y_1 + Y_2)V_y = ?$

<u>Solution:</u> The phasor-form source current I_y can be calculated using Ohm's law and the equivalent admittance seen between the terminals of the current source as

$$I_y = Y_{\text{eq}}V_y = \Bigg(\underbrace{-j0.02}_{Y_1} + \underbrace{0.005\sqrt{3} + j0.015}_{Y_2}\Bigg)\Bigg(\underbrace{e^{j60^\circ}}_{V_y}\Bigg) = \left(0.01e^{-j30^\circ}\ \text{S}\right)\left(e^{j60^\circ}\ \text{V}\right) = \underline{10e^{j30^\circ}\ \text{mA}}.$$



4.d. Equivalent impedance of a parallel *RLC* circuit (optional). In the parallel *RLC* circuit shown, the element values are given by $R = 8\,\Omega$, $L = 2$ mH, and $C = 5\,\mu$F respectively. Find the equivalent impedance of this circuit at three different frequencies: $\omega_1 = 5{,}000$ rad/s, $\omega_2 = 10{,}000$ rad/s, and $\omega_3 = 20{,}000$ rad/s.

<u>Solution:</u> Since the three impedances are connected in parallel, the equivalent impedance Z_{eq} seen between terminals A and B at a signal frequency ω is given by

$$Z_{\text{eq}}(\omega) = \frac{1}{Z_R^{-1} + \underbrace{Z_L^{-1}(\omega)}_{-j/(\omega L)} + \underbrace{Z_C^{-1}(\omega)}_{j\omega C}} = \frac{1}{R^{-1} + j\left(\omega C - \dfrac{1}{\omega L}\right)} = \frac{1}{0.125 + j\left(5\times10^{-6}\omega - 500/\omega\right)}.$$

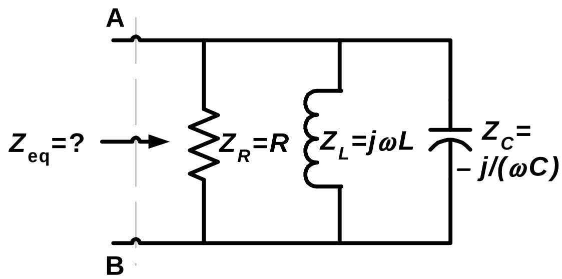
Substituting $\omega_1 = 5{,}000$ rad/s, $\omega_2 = 10{,}000$ rad/s, and $\omega_3 = 20{,}000$ rad/s, we calculate the equivalent impedance Z_{eq} at each frequency as

$$Z_{\text{eq}}(\omega_1) = \frac{1}{0.125 - j0.075} = \frac{0.125 + j0.075}{(0.125)^2 + (0.075)^2} \cong \underline{(5.88 + j3.53)\,\Omega}\,,$$

$$Z_{\text{eq}}(\omega_2) = \frac{1}{0.125} = \underline{8\,\Omega}\text{, and}$$

$$Z_{\text{eq}}(\omega_3) = \frac{1}{0.125 + j0.075} = \frac{0.125 - j0.075}{(0.125)^2 + (0.075)^2} \cong \underline{(5.88 - j3.53)\,\Omega}$$

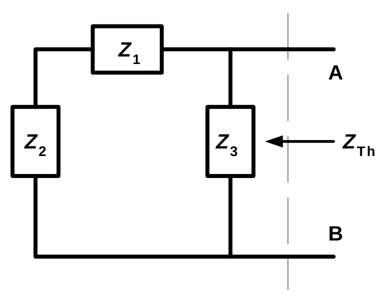
respectively.



4.k. Thevenin impedance (optional). For the impedance circuit shown below, if $Z_1 = (2 - j2)\,\Omega$, $Z_2 = (j2 + 2)\,\Omega$, $Z_3 = 2 - j6\,\Omega$, what is the Thevenin impedance Z_{Th} ?

<u>Solution:</u> The Thevenin impedance seen between terminals A and B can be calculated as

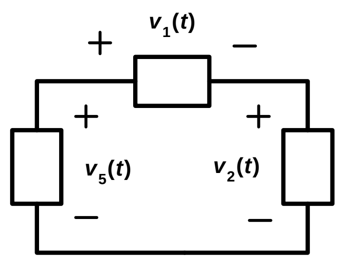
$$Z_{\text{Th}} = \frac{Z_3(Z_1 + Z_2)}{Z_1 + Z_2 + Z_3} = \frac{(2 - j6)(2 - j2 + j2 + 2)}{2 - j2 + j2 + 2 + 2 - j6} = \frac{4(2 - j6)}{6 - j6} = \frac{4(1 - j3)(1 + j)}{\underbrace{3(1 - j)(1 + j)}_{3(1+1)}} = \underline{\left(\frac{8}{3} - j\frac{4}{3}\right)\Omega}\,.$$



P 5. Addition/subtraction of sinusoidal signals. Four sinusoidal-voltage signals are given by $v_1(t) = 10\sin(10^5 t + \pi/2)$, $v_2(t) = 10\cos(10^5 t - 2\pi/3)$, $v_3(t) = 10\sin(10^5 t + \pi/6)$, and $v_4(t) = 10\cos(10^5 t - \pi)$ respectively. Find each of the following signals below and express them in terms of a single sinusoidal waveform: (<u>Suggestion:</u> Use the phasor-domain approach to obtain the above voltages.)

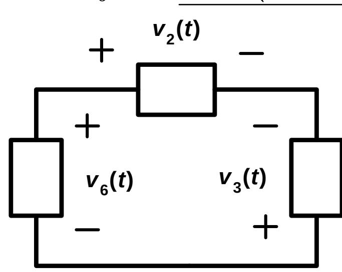
5.a. $v_5(t) = v_1(t) + v_2(t) = ?$

Solution: Using the phasor-form of the time-domain voltage signals $v_1(t)$ and $v_2(t)$ given by $V_1 = 10e^{j0} = 10$ and $V_2 = 10e^{-j2\pi/3} = -5 - j5\sqrt{3}$, the phasor-form of the voltage signal $v_5(t)$ can be found from KVL applied around the closed loop as $V_5 = V_1 + V_2 = 10 - 5 - j5\sqrt{3} = 5\left(1 - j\sqrt{3}\right) = 10e^{-j\pi/3}$. Therefore, the time-domain voltage signal $v_5(t)$ is given by $v_5(t) = \underline{10\cos\left(10^5 t - \pi/3\right)}$.



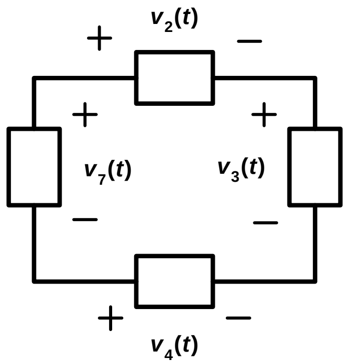
5.b. $v_6(t) = v_2(t) - v_3(t) = ?$

Solution: Using the phasor-form of the time-domain voltage signals $v_2(t)$ and $v_3(t)$ given by $V_2 = 10e^{-j2\pi/3} = -5 - j5\sqrt{3}$ and $V_3 = 10e^{-j\pi/3} = 5 - j5\sqrt{3}$, the phasor-form of the time-domain voltage signal $v_6(t)$ can be found from KVL applied around the closed loop as $V_6 = V_2 - V_3 = -5 - j5\sqrt{3} - 5 + j5\sqrt{3} = -10 = 10e^{j\pi}$. Therefore, the time-domain voltage signal $v_6(t)$ is given by $v_6(t) = \underline{10\cos\left(10^5 t + \pi\right) = -10\cos\left(10^5 t\right)}$.



5.c. $v_7(t) = v_2(t) + v_3(t) - v_4(t) = ?$

Solution: Using the phasor-form of the time-domain voltage signals $v_2(t)$, $v_3(t)$ and $v_4(t)$ given by $V_2 = 10e^{-j2\pi/3} = -5 - j5\sqrt{3}$, $V_3 = 10e^{-j\pi/3} = 5 - j5\sqrt{3}$ and $V_4 = 10e^{-j\pi} = -10$, the phasor-domain voltage V_7 can be found from KVL as $V_7 = V_2 + V_3 - V_4 = -5 - j5\sqrt{3} + 5 - j5\sqrt{3} + 10 = 10 - j10\sqrt{3} = 20e^{-j\pi/3}$. Therefore, the time-domain voltage signal $v_7(t)$ is given by $v_7(t) = \underline{20\cos\left(10^5 t - \pi/3\right)}$.



5.d. (Optional.) $v_8(t) = v_1(t) - v_2(t) + v_3(t) = ?$

<u>Solution:</u> Using the phasor-form of the voltage signals $v_1(t)$, $v_2(t)$ and $v_3(t)$ given by $V_1 = 10e^{j0} = 10$, $V_2 = 10e^{-j2\pi/3} = -5 - j5\sqrt{3}$ and $V_3 = 10e^{-j\pi/3} = 5 - j5\sqrt{3}$, the phasor voltage V_8 is found as $V_8 = V_1 + V_3 - V_2 = 10 + 5 - j5\sqrt{3} + 5 + j5\sqrt{3} = 20 = 20e^{j0}$. Therefore, the time-domain voltage signal $v_8(t)$ is given by $v_8(t) = \underline{20\cos(10^5 t)}$.

5.e. (Optional.) $v_9(t) = v_2(t) - v_3(t) + v_4(t) = ?$

<u>Solution:</u> Using the phasor-form of the voltage signals $v_2(t)$, $v_3(t)$ and $v_4(t)$ given by $V_2 = 10e^{-j2\pi/3} = -5 - j5\sqrt{3}$, $V_3 = 10e^{-j\pi/3} = 5 - j5\sqrt{3}$ and $V_4 = 10e^{-j\pi} = -10$, the phasor voltage V_9 is found as $V_9 = V_2 + V_4 - V_3 = -5 - j5\sqrt{3} - 10 - 5 + j5\sqrt{3} = -20 = 20e^{j\pi}$. Therefore, the time-domain voltage signal $v_9(t)$ is given by $v_9(t) = \underline{20\cos(10^5 t + \pi) = -20\cos(10^5 t)}$.

5.f. (Optional.) $v_{10}(t) = v_1(t) - v_3(t) - v_4(t) = ?$

<u>Solution:</u> Using the phasor-form of the voltage signals $v_1(t)$, $v_3(t)$ and $v_4(t)$ given by $V_1 = 10e^{j0} = 10$, $V_3 = 10e^{-j\pi/3} = 5 - j5\sqrt{3}$ and $V_4 = 10e^{-j\pi} = -10$, the phasor voltage V_{10} is found as $V_{10} = V_1 - V_3 - V_4 = 10 - 5 + j5\sqrt{3} + 10 = 15 + j5\sqrt{3} = 10\sqrt{3}e^{j\pi/6}$. Therefore, the time-domain signal $v_{10}(t)$ is given by $v_{10}(t) = \underline{10\sqrt{3}\cos(10^5 t + \pi/6)}$.

<u>Another important reminder:</u>

<u>EE 261–Final Exam is scheduled for Wednesday 14, 2011, 13:30-15:00!</u>

It is a 90 minute closed book exam. Formula sheets are allowed.