University of Portland  
School of Engineering  

EE 261  
Spring 2011  
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Solutions to Homework # 7—  
Arithmetic of Complex Numbers and Phasor Analysis  
(Wednesday, April 27, 2011)  

P 1. Rectangular to polar form. Write the following rectangular-form complex numbers in polar form:  

1.a. $V_1 = 5\sqrt{2} + j5\sqrt{2} = 10e^{j45^\circ} \text{ or } 10e^{j(\pi/4)} \text{(in radians)}$  
1.b. $Z_2 = 120j + 50 \equiv 130e^{j67.4^\circ} \text{ or } 130e^{j0.374\pi}$  
1.c. $Y_3 = j0.02 = 0.02e^{j90^\circ} \text{ or } 0.02e^{j\pi/2}$  
1.d. $I_4 = -10 - j10\sqrt{3} = 20e^{-j120^\circ} \text{ or } 20e^{-j2\pi/3}$  
1.e. $Z_5 = -j6,000 + 8,000 \equiv 10,000e^{-j36.87^\circ} \text{ or } 10,000e^{-j0.205\pi}$  
1.f. $V_6 = -2.5 = 2.5e^{j(\pm 180^\circ)} \text{ or } 2.5e^{j(\pm \pi)}$  

P 2. Polar to rectangular form. Convert the following polar-form complex numbers in rectangular form:  

2.a. $I_1 = 10e^{j180^\circ} = -10 + j0 = -10$  
2.b. $V_2 = 3\sqrt{2}e^{-j45^\circ} = 3 - j3$  
2.c. $Z_3 = 200e^{-j90^\circ} = 0 - j200 = -j200$  
2.d. $Y_4 = \left(e^{-j2\pi/3}\right)/500 = (1/500)(-1/2 - j\sqrt{3}/2) = -0.001 - j0.001\sqrt{3}$  
2.e. $V_5 = 2.3e^{-j5\pi/6} = 2.3\left(-\sqrt{3}/2 - j/2\right) = -1.15\sqrt{3} - j1.15$  
2.f. $I_6 = 5.6\times10^{-3}e^{j240^\circ} = 5.6\times10^{-3}\left(-1/2 - j\sqrt{3}/2\right) = -2.8\times10^{-3} - j2.8\times10^{-3}\sqrt{3}$
P 3. Phasor representation of sinusoidal signals. Find the phasor-form representation of the following sinusoidal signals. (Note that the capital letters represent the phasor form.)

3.a. \( i_1(t) = 2 \times 10^{-4} \cos(2\pi \times 10^4 t + 60^\circ) \rightarrow I_1 = 2 \times 10^{-4} e^{j60^\circ} \)

3.b. \( v_2(t) = 3.8 \cos(10^5 t - 25^\circ) \rightarrow V_2 = 3.8e^{-j25^\circ} \)

3.c. \( v_a(t) = 1.73 \sin(10^5 \pi + \pi/10) \rightarrow V_a = 1.73e^{j(\pi/10 - \pi/2)} = 1.73e^{-j\pi/5} \)

3.d. \( i_c(t) = -4.2 \sin(3\pi \times 10^5 t - 3\pi/5) \rightarrow I_c = 4.2e^{j(3\pi/2 - 3\pi/5)} = 4.2e^{-j\pi/10} \)

3.e. \( i_s(t) = 0.1 \cos(6.28 \times 10^4 t - 135^\circ) \rightarrow I_s = 0.1e^{-j135^\circ} \)

3.f. \( v_s(t) = 10 \sin(8\pi \times 10^4 t - 300^\circ) \rightarrow V_s = 10e^{j(300^\circ - 90^\circ)} = 10e^{-j30^\circ} \)

3.g. \( i_t(t) = 0.03 \cos(10^5 t - 135^\circ) - 0.015 \sin(10^5 t + 60^\circ) \)

\[ I_t = 0.03e^{-j135^\circ} - 0.015e^{-j130^\circ} \]

3.h. \( v_t(t) = 2.1 \sin(4.4\pi \times 10^4 t - 30^\circ) + 4.9 \cos(4.4\pi \times 10^4 t + 30^\circ) \)

\[ V_t = 2.1e^{j30^\circ} + 4.9e^{-j120^\circ} \]

P 4. Basic arithmetic operations of complex numbers. Solve the following circuit problems, simplify each answer and provide it in polar form.

4.a. Kirchhoff’s current law (KCL) applied in the phasor domain. Given \( I_1 = 10e^{j30^\circ}, I_2 = 10e^{-j30^\circ}, I_3 = I_1 + I_2 = ? \) (Note that each \( I_k \) current is a phasor quantity which represents a real-time Sinusoidal Steady-State (SSS) current \( i_k(t) \) flowing in the circuit shown below. Just like the time-domain currents, the phasor-domain currents must also satisfy KCL.)

**Solution:** Transforming polar-form phasor currents into rectangular form, we can find the phasor current \( I_3 \) as

\[ I_3 = I_1 + I_2 = 10 \left( \frac{\sqrt{3}}{2} + j \frac{1}{2} \right) + 10 \left( \frac{\sqrt{3}}{2} - j \frac{1}{2} \right) = 10\sqrt{3} + j0 = 10\sqrt{3}e^{j0} \].

4.b. Kirchhoff’s voltage law (KVL) in phasor form. In the phasor-domain circuit shown below, \( V_1 = 5\sqrt{2}e^{-j45^\circ}, V_2 = 5\sqrt{2}e^{j45^\circ}, V_3 = V_1 - V_2 = ? \) (Note that each \( V_k \) is a phasor quantity which represents a real-time SSS voltage \( v_k(t) \) in the circuit shown below. The phasor-domain voltages must satisfy KVL.)
**Solution:** Using rectangular-form phasors $V_1$ and $V_2$, the phasor voltage $V_3$ can be found as $V_3 = V_1 - V_2 = 5\sqrt{2} \left( \frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}} \right) - 5\sqrt{2} \left( \frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}} \right) = 0 - j10 = 10e^{-j90^\circ}$.

![Phasor Diagram](image)

**4.c. Equivalent impedance.** The circuit shown between terminals A and B is shown in phasor domain. If $Z_1 = (200 - 300j)\Omega$ and $Z_2 = (100 + j200)\Omega$, $Z_3 = Z_1 + Z_2 = \Omega$ (Note that each Z represent an impedance. Impedances can be combined in series or in parallel just like resistances.)

**Solution:** Since the two impedances are connected in series, the equivalent impedance $Z_{eq}$ is given by $Z_{eq} = Z_1 + Z_2 = (200 - 300j) + (100 + j200) = 300 - j100 \equiv 316e^{-j18.43^\circ} \Omega$.

![Equivalent Impedance Diagram](image)

**4.d. Equivalent impedance of a series RLC circuit.** In the series RLC circuit shown, the element values are given by $R = 4 \Omega$, $L = 5$ mH, and $C = 1.25$ mF respectively. Find the equivalent impedance of this circuit at three different frequencies: $\omega_1 = 200$ rad/s, $\omega_2 = 400$ rad/s, and $\omega_3 = 1,200$ rad/s.

**Solution:** Since the three impedances are connected in series, the equivalent impedance $Z_{eq}$ at any frequency is given by $Z_{eq}(\omega) = Z_R + Z_L(\omega) + Z_C(\omega) = 4 + j \left( 5\times10^{-3} - \frac{800}{\omega} \right)$.

Substituting the frequency values $\omega_1 = 200$ rad/s, $\omega_2 = 400$ rad/s, and $\omega_3 = 1,200$ rad/s, we find $Z_{eq}(\omega_1) = (4 - j3)\Omega$, $Z_{eq}(\omega_2) = 4 \Omega$, and $Z_{eq}(\omega_3) = (4 + j5.33)\Omega$ respectively.
4.e. Equivalent admittance. Admittance of an element represented by \( Y \) (in Siemens) is defined as the inverse of the impedance \( Z \) (in \( \Omega \)) of the same element, i.e., \( Y = Z^{-1} \).

Given \( Y_1 = 0.002e^{j\pi/2} \) S, \( Y_2 = 0.002\sqrt{2}e^{-j\pi/4} \) S, what is \( Y_3 = Y_1 + Y_2 = ? \) (Note that admittances can be combined in series or in parallel just like the same way as conductances.)

**Solution:** Since the two admittances are connected in parallel, the equivalent admittance \( Y_{eq} \) is given by \( Y_{eq} = Y_1 + Y_2 = 0 + j0.002 + 0.002 - j0.002 = 0.002 \equiv 0.002e^{j0} \) S.

4.f. Ohm’s law in phasor form. Given \( I_1 = 0.02e^{-j\pi/3} \) A, \( Z_1 = 150e^{j\pi/6} \) \( \Omega \), \( V_1 = Z_1I_1 = ? \) (Note that \( V = ZI \) is the phasor-domain equivalent of the time-domain Ohm’s law given as \( v(t) = Ri(t) \). Note also that Ohm’s law in phasor form is not only limited to resistors but can also be used for inductors and capacitors.)

**Solution:** Using Ohm’s law in phasor form, the phasor voltage \( V_1 \) can be obtained as \( V_1 = (150e^{j\pi/6})(0.02e^{-j\pi/3}) = 3e^{-j\pi/6} \) V.
4.g. Phasor-domain solution of sinusoidal steady-state circuits. In the circuit shown, given $V_s = 4e^{j30^\circ}$ V, $I_1 = 0.02e^{-j23.13^\circ}$ A, $Z_1 = (40 + j80)\Omega$, $Z_2 = (30 - j20)\Omega$, $Z_3 = ?$

**Solution:** Using Ohm’s law in phasor form, we can find the equivalent impedance $Z_{eq}$ seen from the source as $Z_{eq} = V_s/I_1 = 4e^{j30^\circ}/0.02e^{-j23.13^\circ} = 200e^{j33.13^\circ} = 120 + j160\Omega$. However, since the three impedances are connected in series, $Z_{eq}$ is also given by $Z_{eq} = Z_1 + Z_2 + Z_3 = (40 + j80) + (30 - j20) + Z_3 = 120 + j160\Omega$ from which we can obtain the unknown impedance $Z_3$ as $Z_3 = (50 + j100)\Omega$.

4.h. Thevenin impedance. For the phasor-domain circuit shown, given the three impedance values to be $Z_1 = 50\Omega$, $Z_2 = j50\Omega$, $Z_3 = -j50\Omega$, $Z_{Th} = Z_3 + \frac{Z_1Z_2}{Z_1 + Z_2} = ?$

**Solution:** The Thevenin impedance seen between terminals A and B can be calculated as $Z_{Th} = -j50 + \frac{(50)(j50)}{50 + j50} = -j50 + \frac{j50}{1 + j} = -j50 + \frac{j50(1 - j)}{(1 + j)(1 - j)} = -j50 + 25 + j25 = 25 - j25\Omega$.

4.i. KVL and Ohm’s law. In the phasor-domain circuit shown below, given $Z_1 = (300 + j300)\Omega$, $Z_2 = 300\sqrt{2}e^{-j\pi/4} \Omega$, $I_x = 0.04e^{j\pi/3}$ A, $V_x = (Z_1 + Z_2)I_x = ?$

**Solution:** The phasor-form source voltage $V_x$ can be calculated using Ohm’s law and the equivalent impedance seen between the terminals of the voltage source as $V_x = Z_{eq}I_x = \left(\frac{300 + j300 + 300 - j300}{Z_1 + Z_2}\right)(0.04e^{j\pi/3}) = (600\Omega)(0.04e^{j\pi/3} A) = 24e^{j\pi/3} V$. 

![Diagram](image-url)
4.j. **KCL and Ohm’s law (optional).** In the following phasor-domain circuit shown, 
\[ Y_1 = 0.02e^{-j\pi/2} \, \text{S}, \quad Y_2 = 0.01\sqrt{3}e^{j\pi/3} \, \text{S}, \quad V_y = e^{j60^\circ} \, \text{V}, \quad I_y = (Y_1 + Y_2)V_y = ? \]

**Solution:** The phasor-form source current \( I_y \) can be calculated using Ohm’s law and the equivalent admittance seen between the terminals of the current source as
\[
I_y = Y_{eq}V_y = \left( -j0.02 + 0.005\sqrt{3} + j0.015 \right) \left( e^{j60^\circ} / V_y \right) = \left( 0.01e^{-j30^\circ} \cdot e^{j60^\circ} \right) \text{V} = 10e^{j30^\circ} \text{mA}.
\]

4.d. **Equivalent impedance of a parallel RLC circuit (optional).** In the parallel RLC circuit shown, the element values are given by \( R = 8 \, \Omega, \quad L = 2 \, \text{mH}, \quad \text{and} \, C = 5 \, \mu\text{F} \) respectively. Find the equivalent impedance of this circuit at three different frequencies: \( \omega_1 = 5,000 \, \text{rad/s}, \quad \omega_2 = 10,000 \, \text{rad/s}, \quad \text{and} \, \omega_3 = 20,000 \, \text{rad/s}. \)

**Solution:** Since the three impedances are connected in parallel, the equivalent impedance \( Z_{eq} \) seen between terminals A and B at a signal frequency \( \omega \) is given by
\[
Z_{eq}(\omega) = \frac{1}{\frac{1}{Z_R^{-1}} + \frac{1}{Z_L^{-1}(\omega)} + \frac{1}{Z_C^{-1}(\omega)}} = \frac{1}{R^{-1} + j\left( \omega C - \frac{1}{\omega L} \right)} = 0.125 + j\left( 5 \times 10^{-6} \omega - 500/\omega \right).
\]

Substituting \( \omega_1 = 5,000 \, \text{rad/s}, \quad \omega_2 = 10,000 \, \text{rad/s}, \quad \text{and} \, \omega_3 = 20,000 \, \text{rad/s}, \) we calculate the equivalent impedance \( Z_{eq} \) at each frequency as.
\[ Z_{eq}(\omega_1) = \frac{1}{0.125 - j0.075} = \frac{0.125 + j0.075}{(0.125)^2 + (0.075)^2} = (5.88 + j3.53)\Omega, \]

\[ Z_{eq}(\omega_2) = \frac{1}{0.125} = 8\Omega, \text{ and } \]

\[ Z_{eq}(\omega_3) = \frac{1}{0.125 + j0.075} = \frac{0.125 - j0.075}{(0.125)^2 + (0.075)^2} = (5.88 - j3.53)\Omega \]

respectively.

4.k. **Thevenin impedance (optional).** For the impedance circuit shown below, if \( Z_1 = (2 - j2)\Omega \), \( Z_2 = (j2 + 2)\Omega \), \( Z_3 = 2 - j6\Omega \), what is the Thevenin impedance \( Z_{\text{th}} \)?

**Solution:** The Thevenin impedance seen between terminals A and B can be calculated as

\[ Z_{\text{th}} = \frac{Z_1(Z_1 + Z_2)}{Z_1 + Z_2 + Z_3} = \frac{(2 - j6)(2 - j2 + j2 + 2)}{2 - j2 + j2 + 2 + 2 - j6} = \frac{4(2 - j6)}{6 - j6} = \frac{4(1 - j3)(1 + j)}{3(1 - j3)(1 + j)} = \left( \frac{8 - j4}{3 - \frac{4}{3}} \right)\Omega. \]

**P 5. Addition/subtraction of sinusoidal signals.** Four sinusoidal-voltage signals are given by \( v_1(t) = 10\sin(10^5t + \pi/2) \), \( v_2(t) = 10\cos(10^5t - 2\pi/3) \), \( v_3(t) = 10\sin(10^5t + \pi/6) \), and \( v_4(t) = 10\cos(10^5t - \pi) \) respectively. Find each of the following signals below and express them in terms of a single sinusoidal waveform: (Suggestion: Use the phasor-domain approach to obtain the above voltages.)
5.a. \( v_5(t) = v_1(t) + v_2(t) = ? \)

**Solution:** Using the phasor-form of the time-domain voltage signals \( v_1(t) \) and \( v_2(t) \) given by \( V_1 = 10e^{j0} = 10 \) and \( V_2 = 10e^{-j2\pi/3} = -5 - j5\sqrt{3} \), the phasor-form of the voltage signal \( v_5(t) \) can be found from KVL applied around the closed loop as \( V_5 = V_1 + V_2 = 10 - 5 - j5\sqrt{3} = 5(1 - j\sqrt{3}) = 10e^{-j\pi/3} \). Therefore, the time-domain voltage signal \( v_5(t) \) is given by \( v_5(t) = 10\cos(10^5t - \pi/3) \).

![Diagram](image)

5.b. \( v_6(t) = v_5(t) - v_3(t) = ? \)

**Solution:** Using the phasor-form of the time-domain voltage signals \( v_2(t) \) and \( v_5(t) \) given by \( V_2 = 10e^{-j2\pi/3} = -5 - j5\sqrt{3} \) and \( V_3 = 10e^{-j\pi/3} = 5 - j5\sqrt{3} \), the phasor-form of the time-domain voltage signal \( v_6(t) \) can be found from KVL applied around the closed loop as \( V_6 = V_2 - V_3 = -5 - j5\sqrt{3} - 5 + j5\sqrt{3} = -10 = 10e^{j\pi} \). Therefore, the time-domain voltage signal \( v_6(t) \) is given by \( v_6(t) = 10\cos(10^5t + \pi) = -10\cos(10^5t) \).

![Diagram](image)

5.c. \( v_7(t) = v_2(t) + v_3(t) - v_4(t) = ? \)

**Solution:** Using the phasor-form of the time-domain voltage signals \( v_2(t) \), \( v_3(t) \) and \( v_4(t) \) given by \( V_2 = 10e^{-j2\pi/3} = -5 - j5\sqrt{3} \), \( V_3 = 10e^{-j\pi/3} = 5 - j5\sqrt{3} \) and \( V_4 = 10e^{-j\pi} = -10 \), the phasor-domain voltage \( V_7 \) can be found from KVL as \( V_7 = V_2 + V_3 - V_4 = -5 - j5\sqrt{3} + 5 - j5\sqrt{3} + 10 = 10 - j10\sqrt{3} = 20e^{-j\pi/3} \). Therefore, the time-domain voltage signal \( v_7(t) \) is given by \( v_7(t) = 20\cos(10^5t - \pi/3) \).
5.d. (Optional.) $v_8(t) = v_1(t) - v_2(t) + v_3(t) = ?$

**Solution:** Using the phasor-form of the voltage signals $v_1(t)$, $v_2(t)$ and $v_3(t)$ given by $V_1 = 10e^{j0} = 10$, $V_2 = 10e^{-j2\pi/3} = -5 - j5\sqrt{3}$ and $V_3 = 10e^{-j\pi/3} = 5 - j5\sqrt{3}$, the phasor voltage $V_8$ is found as $V_8 = V_1 + V_3 - V_2 = 10 + 5 - j5\sqrt{3} + 5 + j5\sqrt{3} = 20 = 20e^{i0}$.

Therefore, the time-domain voltage signal $v_8(t)$ is given by $v_8(t) = 20\cos(10^5 t)$.

5.e. (Optional.) $v_9(t) = v_2(t) - v_3(t) + v_4(t) = ?$

**Solution:** Using the phasor-form of the voltage signals $v_2(t)$, $v_3(t)$ and $v_4(t)$ given by $V_2 = 10e^{-j2\pi/3} = -5 - j5\sqrt{3}$, $V_3 = 10e^{-j\pi/3} = 5 - j5\sqrt{3}$ and $V_4 = 10e^{-j\pi} = -10$, the phasor voltage $V_9$ is found as $V_9 = V_3 + V_4 - V_2 = -5 - j5\sqrt{3} - 10 - 5 + j5\sqrt{3} = -20 = 20e^{j\pi}$.

Therefore, the time-domain voltage signal $v_9(t)$ is given by $v_9(t) = 20\cos(10^5 t + \pi) = -20\cos(10^5 t)$.

5.f. (Optional.) $v_{10}(t) = v_1(t) - v_3(t) - v_4(t) = ?$

**Solution:** Using the phasor-form of the voltage signals $v_1(t)$, $v_3(t)$ and $v_4(t)$ given by $V_1 = 10e^{j0} = 10$, $V_3 = 10e^{-j\pi/3} = 5 - j5\sqrt{3}$ and $V_4 = 10e^{-j\pi} = -10$, the phasor voltage $V_{10}$ is found as $V_{10} = V_1 - V_3 - V_4 = 10 - 5 + j5\sqrt{3} + 10 = 15 + j5\sqrt{3} = 10\sqrt{3}e^{j\pi/6}$.

Therefore, the time-domain signal $v_{10}(t)$ is given by $v_{10}(t) = 10\sqrt{3}\cos(10^5 t + \pi/6)$.

**Another important reminder:**

EE 261 – Final Exam is scheduled for Thursday, May 5, 2011, 8:30-10:00 a.m.!

It is a 90 minute closed book exam. Formula sheets are allowed.