EE 262
SPRING 2012
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SOLUTIONS TO HOMEWORK #5

(#1) (a) $h(t) = \frac{dy_s(t)}{dt}$

I'm the unit-step response & someone is taking my derivative!

I'm the impulse response!

$$= \left[-5e^{-t}(3+2t) + 10e^{-t}\right]u(t) + 5e^{-t}(3+2t)\delta(t)$$

$$= (-5e^{-t} - 10te^{-t})u(t) + 15\delta(t)$$

$$= \boxed{15\delta(t) - 5e^{-t}(1+2t)u(t)}$$

(b) $y(t) = 3y_s(t-1) - 4h(t-2)$

$$= 15e^{-(t-1)}(3+2(t-1))u(t-1) - 4\left[15\delta(t-2) - 5e^{-(t-2)}(1+2(t-2))u(t-2)\right]$$

$$= \boxed{15e^{-(t-1)}(1+2t)u(t-1) + 20e^{-(t-2)}(2t-3)u(t-2) - 60\delta(t-2)}$$

(#2) The unit-step response of the LTI system can be obtained as follows:

$$y_s[n] = \sum_{k=0}^{\infty} h[n-k] = \sum_{k=0}^{\infty} 2^{-(n-k)+1} u[n-k]$$

I'm the unit-step response of the DT-LTI system!

I'm zero when $k > n$!

And 1 when $k \leq n$!

$$= 2^{1-n} \sum_{k=0}^{n} 2^k$$

Using $y_s[n]$ obtained, $y[n]$ due to $x[n] = 3u[n-1]$ can be found as $\boxed{y[n] = 6(2)^{-n} \sum_{k=0}^{n-1} 2^k}$

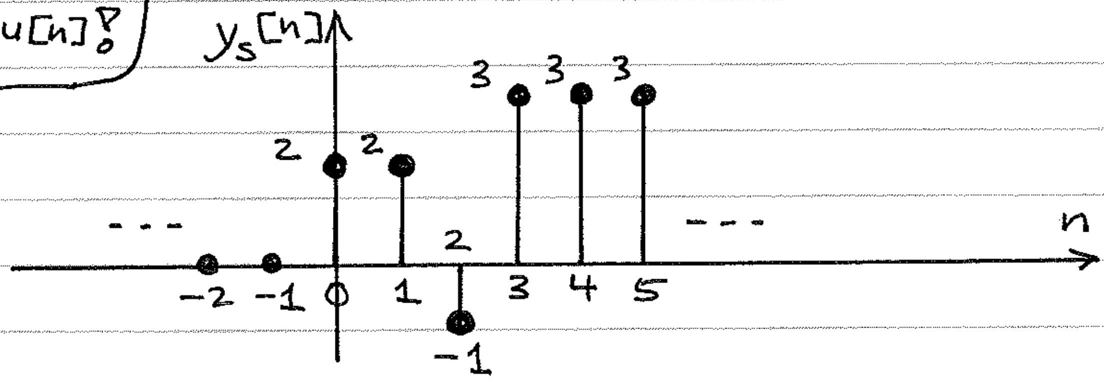
(#3) (a) $x[n] = u[n]$

I'm the zero-state response due to $x[n]=u[n]$!

$$y_s[n] = x[n] * h[n]$$

$$= u[n] * \left(2\delta[n] - 3\delta[n-2] + 4\delta[n-3]\right)$$

$$= \boxed{2u[n] - 3u[n-2] + 4u[n-3]}$$



(b) $x[n] = 2u[n] - 3u[n-1]$

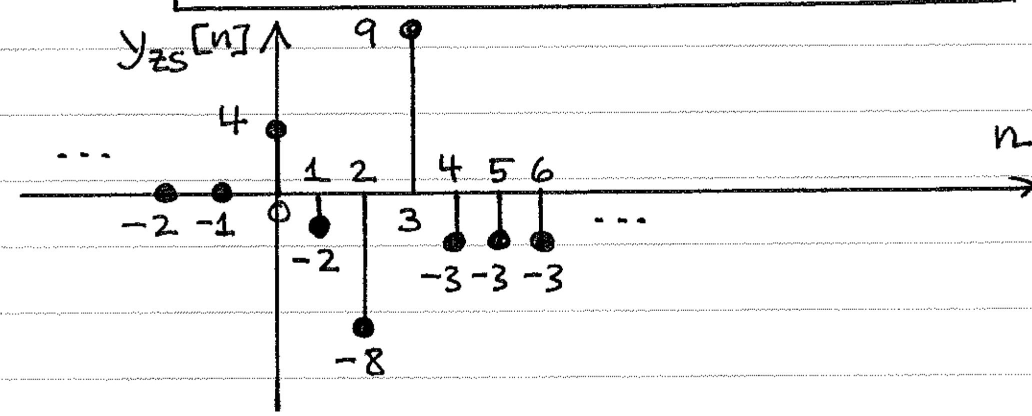
I'm the zero-state response!

$$y_{zs}[n] = 2y_s[n] - 3y_s[n-1]$$

$$= 4u[n] - 6u[n-2] + 8u[n-3]$$

$$-6u[n-1] + 9u[n-3] - 12u[n-4]$$

$$= \boxed{4u[n] - 6u[n-1] - 6u[n-2] + 17u[n-3] - 12u[n-4]}$$



(#4) (a) Characteristic equation is given by

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$\rightarrow s^2 + 2s + 1 = 0 \rightarrow (s+1)^2 = 0$$

∴ Critically-damped response.

$$y_{1s}(t) = (A_1 + A_2 t)e^{-t} + A_3$$

$$y_{1s}(0) = A_1 + A_3 = 0$$

$$y_{1s}(t \rightarrow \infty) = A_3 = 1 \rightarrow A_1 = -A_3 = -1$$

$$y_{2s}(t) = C\frac{dy_{1s}(t)}{dt} = 0.2\left[A_2 e^{-t} - (A_1 + A_2 t)e^{-t}\right]$$

$$y_{2s}(0) = 0.2\left[A_2 - A_1\right] = 0 \rightarrow A_2 = A_1 = -1$$

I'm the unit-step response!

$$\therefore\ y_{1s}(t) = \boxed{\left[-(1+t)e^{-t} + 1\right]u(t)}$$

Me too!

$$y_{2s}(t) = C\frac{dy_{1s}(t)}{dt} = 0.2\left\{\left[-e^{-t} + (1+t)e^{-t}\right]u(t) + \underbrace{\left[-(1+t)e^{-t} + 1\right]}_{=0 \text{ at } t=0}\delta(t)\right\}$$

$$= \boxed{0.2\,te^{-t}u(t)}$$

(b) $$h_1(t) = \frac{dy_{1s}(t)}{dt} = \boxed{te^{-t}u(t)}$$

We are impulse responses!

$$h_2(t) = \frac{dy_{2s}(t)}{dt} = \left(0.2e^{-t} - 0.2te^{-t}\right)u(t) + \underbrace{0.2te^{-t}}_{=0 \text{ at } t=0}\delta(t)$$

I'm a transfer function!

$$= \boxed{0.2(1-t)e^{-t}u(t)}$$

(c) $$H_1(s) = \int_{-\infty}^{\infty} h_1(t)e^{-st}\,dt = \int_0^{\infty} t\underbrace{e^{-t}e^{-st}}_{e^{-(s+1)t}}\,dt$$

Using integration by parts ($u = t$ & $dV = e^{-(s+1)t}dt$):

$$H_1(s) = \left[-\frac{te^{-(s+1)t}}{(s+1)}\right]_0^{\infty} + \frac{1}{s+1}\int_0^{\infty} e^{-(s+1)t}\,dt$$

$$= \frac{1}{s+1}\left[-\frac{e^{-(s+1)t}}{(s+1)}\right]_0^{\infty} = \boxed{\frac{1}{(s+1)^2}}$$

Me too!

$$H_2(s) = \int_{-\infty}^{\infty} h_2(t)e^{-st}\,dt = 0.2\int_0^{\infty} \underbrace{(1-t)}_{u}\underbrace{e^{-t}e^{-st}\,dt}_{dV}$$

$$= 0.2\left[(1-t)\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^{\infty} - 0.2\int_0^{\infty}\frac{e^{-(s+1)t}}{s+1}\,dt$$

$$= \frac{0.2}{s+1} + \left[\frac{0.2e^{-(s+1)t}}{(s+1)^2}\right]_0^{\infty}$$

$$= \frac{0.2}{s+1} - \frac{0.2}{(s+1)^2} = \boxed{\frac{0.2s}{(s+1)^2}}$$

(d) $H_1(\omega) = H_1(s)\Big|_{s=\hat{j}\omega} = \dfrac{1}{(\hat{j}\omega+1)^2}$

We are frequency responses!

We both have magnitude and phase angle!

$$= \boxed{\frac{1}{1+\omega^2}\, e^{-\hat{j}2\tan^{-1}\omega}}$$

$H_2(\omega) = H_2(s)\Big|_{s=\hat{j}\omega}$

I'm an impulse response!

$$= \boxed{\frac{0.2\,\omega}{1+\omega^2}\, e^{\hat{j}(\frac{\pi}{2}-2\tan^{-1}\omega)}}$$

(#5) $h_1[n] = \frac{1}{3}u[n]$

I'm a unit-step response!

$y_{1s}[n] = u[n] * h_1[n] = u[n] * \frac{1}{3}u[n] = \boxed{\frac{1}{3}r[n]}$

$y_{2s}[n] = (3-(1/3)^n)\,u[n]$

I'm also an impulse response!

$h_2[n] = y_{2s}[n] - y_{2s}[n-1]$

I'm the overall impulse response!

$$= \boxed{(3-(1/3)^n)\,u[n] - (3-(1/3)^{n-1})\,u[n-1]}$$

(#6) (a) $h_0(t) = h_1(t) * h_2(t)$

$$= 4\cancel{e^2}e^{-t}u(t) * 3\cancel{e^{2}}e^{-2(t-1)}u(t-1)$$

Refer to solution of Problem #1 in Homework #4

$$= 12\,\frac{e^{-(t-1)} - e^{-2(t-1)}}{(2-1)}\,u(t-1)$$

$$= \boxed{12\left[e^{-(t-1)} - e^{-2(t-1)}\right]u(t-1)}$$

I'm the overall impulse response!

(b) $h_0(t) = h_1(t) + h_2(t) = \boxed{4e^{2-t}u(t) + 3e^{-2t}u(t-1)}$

(#7) (a) $h[n] + 0.5h[n-1] = 3\delta[n]$

$n=0 \rightarrow h[0] + 0.5\cancelto{0}{h[-1]} = 3\cancelto{1}{\delta[0]} = 3 \rightarrow h[0] = 3$

$n=1 \rightarrow h[1] + 0.5h[0] = 3\delta[1] = 0 \rightarrow h[1] = -\frac{3}{2}$

$n=2 \rightarrow h[2] + 0.5h[1] = 3\delta[2] = 0 \rightarrow h[2] = \frac{3}{4}$

$n=3 \rightarrow h[3] + 0.5h[2] = 3\delta[3] = 0 \rightarrow h[3] = \frac{-3}{8}$

Recognize a pattern!

$\vdots$

I'm the impulse response!

$\therefore h[n] = \boxed{3(-1)^n\left(\frac{1}{2}\right)^n u[n] = 3\left(-\frac{1}{2}\right)^n u[n]}$

(b) $y_s[n] = \sum_{k=0}^{\infty} h[n-k]$

I'm the unit-step response!

$= \boxed{3\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^{n-k} u[n-k]}$

I'm zero for $k > n$!

I'm the overall impulse response!

(c) $h_0[n] = \sum_{k=-\infty}^{\infty} h[n-k]h[k]$

I'm zero for $k<0$ & 1 for $k \geq 0$!

$= 9\sum_{k=-\infty}^{\infty}\left(-\frac{1}{2}\right)^{n-k} u[n-k]\left(-\frac{1}{2}\right)^k u[k]$

$= 9\sum_{k=0}^{n}\left(-\frac{1}{2}\right)^n = 9\left(-\frac{1}{2}\right)^n \sum_{k=0}^{n} 1 = 9\left(-\frac{1}{2}\right)^n (n+1)$

I'm the overall impulse response

$$\therefore \quad h_0[n] = \boxed{9\left(-\frac{1}{2}\right)^n (n+1)\, u[n]}$$

(d) $h_0[n] = \boxed{6\left(-\frac{1}{2}\right)^n u[n]}$

<u>Fun exercise:</u> Show that the impulse response $h_2[n]$ obtained in Problem #5 can also be rewritten as

$$h_2[n] = 3\delta[n] - \left(\frac{1}{3}\right)^n \left(\delta[n] - 2u[n-1]\right)$$

—— FINITO ——