Fourier Series

Example #1

I'm the Fourier series of a periodic signal ☺

Given $x(t) = 3 - 10\cos(200\pi t) + 6\sin(600\pi t)$

(a) Find ω_0, f_0, and T_0.

(b) Express $x(t)$ in complex exponential Fourier series form and sketch its two-sided spectrum.

Solution:

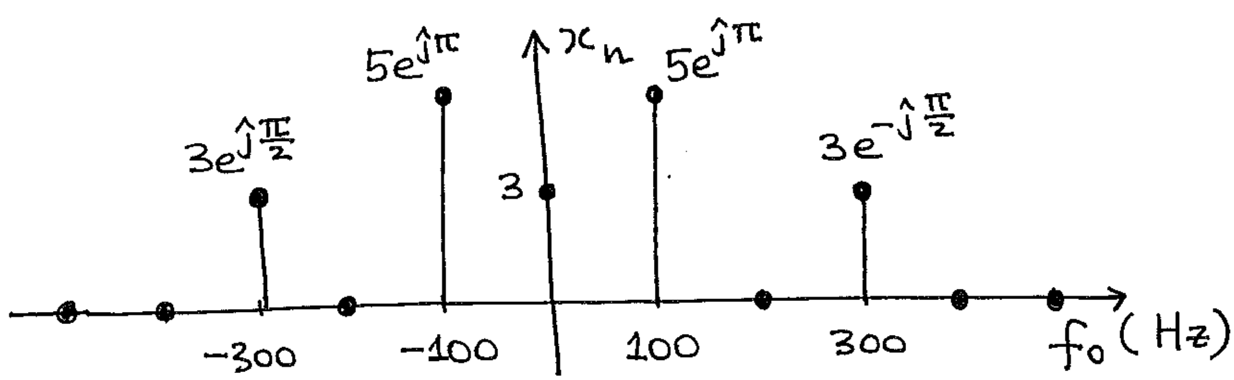
(a) $\omega_0 = 200\pi$ rad/s

$$f_0 = \frac{\omega_0}{2\pi} = 100 \text{ Hz}$$

$$T_0 = f_0^{-1} = 10 \text{ ms}$$

(b)
$$x(t) = 3 - \frac{10e^{\hat{j}200\pi t}}{2} - \frac{10e^{-\hat{j}200\pi t}}{2} + \frac{6e^{\hat{j}600\pi t}}{2\hat{j}} - \frac{6e^{-\hat{j}600\pi t}}{2\hat{j}}$$

$$= 3 + \underbrace{5e^{\hat{j}\pi}}_{x_1} e^{\hat{j}200\pi t} + \underbrace{5e^{\hat{j}\pi}}_{x_{-1}} e^{-\hat{j}200\pi t} + \underbrace{3e^{-\hat{j}\frac{\pi}{2}}}_{x_3} e^{\hat{j}600\pi t} + \underbrace{3e^{\hat{j}\frac{\pi}{2}}}_{x_{-3}} e^{-\hat{j}600\pi t}$$



Example #2

Given $x(t) = -5 - 12\sin(3000\pi t) + 6\cos\left(6000\pi t - \frac{\pi}{4}\right)$

I'm also a Fourier series!

$$- 8\sin\left(12000\pi t + \frac{\pi}{3}\right)$$

(a) Find ω_0, f_0, and T_0.

(b) Express $x(t)$ in complex exponential Fourier series form and sketch its two-sided Fourier spectrum.

Solution:

(a) $\omega_0 = 3000\pi$ rad/s

$$f_0 = \frac{\omega_0}{2\pi} = 1500\,\text{Hz} = 1.5\,\text{kHz}$$

$$T_0 = f_0^{-1} \simeq 6.67 \times 10^{-4}\,\text{s} \simeq 0.667\,\text{ms}$$

(b) Using $e^{\hat{j}\pi} = -1$, $e^{\hat{j}\pi/2} = \hat{j}$, and $e^{-\hat{j}\pi/2} = -\hat{j}$:

$$x(t) = 5e^{\hat{j}\pi} + \frac{12e^{\hat{j}\pi}}{2\hat{j}} e^{\hat{j}3000\pi t} + \frac{12}{2\hat{j}} e^{-\hat{j}3000\pi t}$$

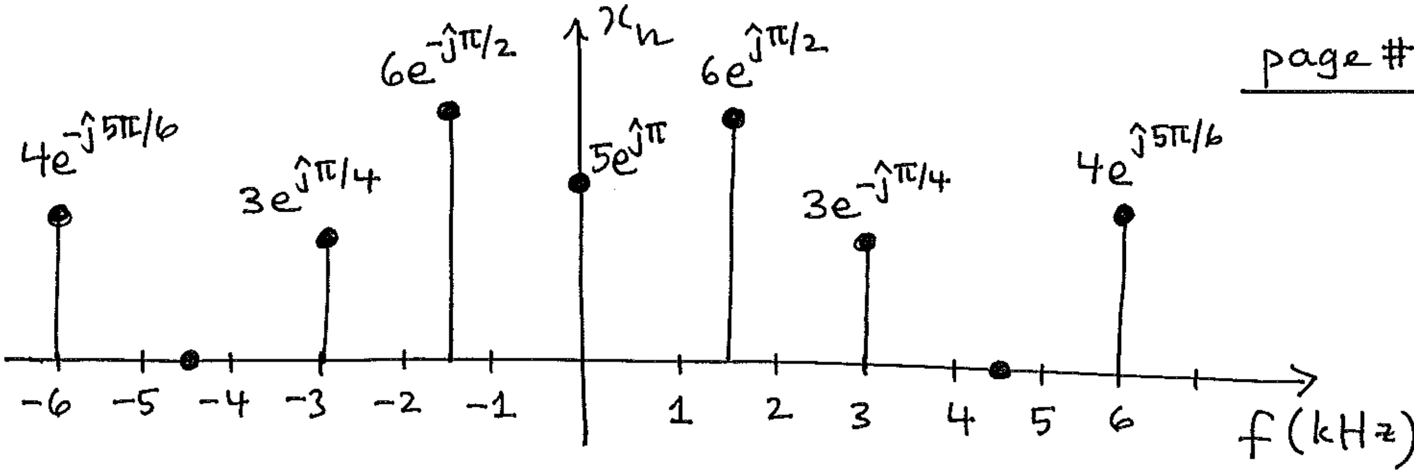
$$+ \frac{6}{2} e^{\hat{j}(6000\pi t - \pi/4)} + \frac{6}{2} e^{-\hat{j}(6000\pi t - \pi/4)}$$

$$+ \frac{8e^{\hat{j}\pi}}{2\hat{j}} e^{\hat{j}(12000\pi t + \pi/3)} + \frac{8}{2\hat{j}} e^{-\hat{j}(12000\pi t + \pi/3)}$$

$$= \underbrace{5e^{\hat{j}\pi}}_{x_0} + \underbrace{6e^{\hat{j}\pi/2}}_{x_1} e^{\hat{j}3000\pi t} + \underbrace{6e^{-\hat{j}\pi/2}}_{x_{-1}} e^{-\hat{j}3000\pi t}$$

$$+ \underbrace{3e^{-\hat{j}\pi/4}}_{x_2} e^{\hat{j}6000\pi t} + \underbrace{3e^{+\hat{j}\pi/4}}_{x_{-2}} e^{-\hat{j}6000\pi t}$$

$$+ \underbrace{4e^{\hat{j}\frac{5\pi}{6}}}_{x_4} e^{\hat{j}12000\pi t} + \underbrace{4e^{-\hat{j}\frac{5\pi}{6}}}_{x_{-4}} e^{-\hat{j}12000\pi t}$$



Example # 3

I'm also the Fourier series of a periodic signal!

Given $x(t) = 4 + 10\cos\left(10^5 t + \frac{2\pi}{3}\right) - 14\sin\left(3\times 10^5 t\right)$

(a) Find ω_0, f_0, and T_0.

(b) Express $x(t)$ in complex exponential Fourier series form and sketch its two-sided spectrum.

Solution:

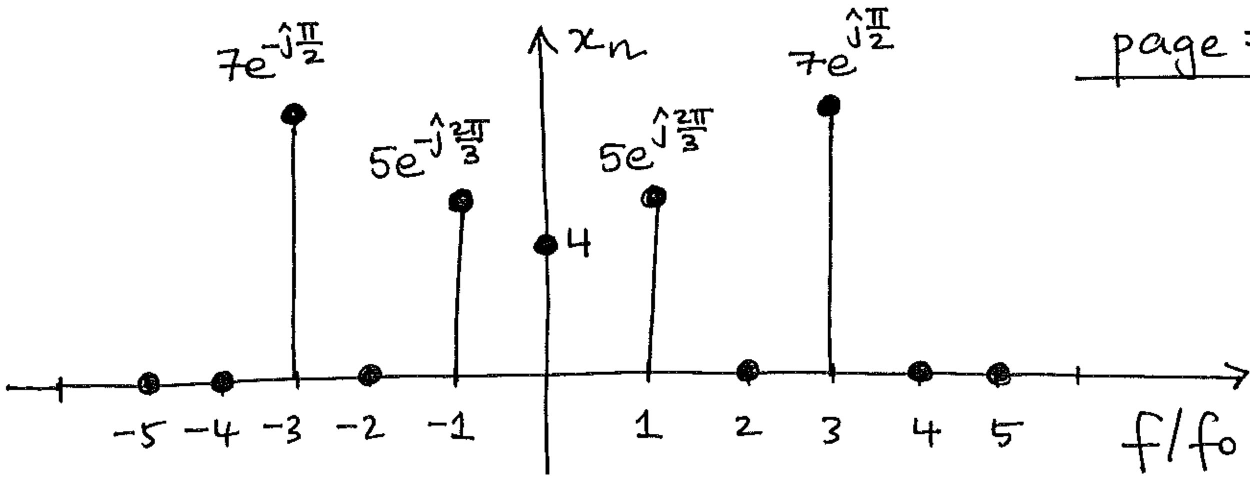
(a) $\omega_0 = 10^5$ rad/s

$$f_0 = \frac{\omega_0}{2\pi} \simeq 15{,}915.5\ \text{Hz} \simeq 15.9\ \text{kHz}$$

$$T_0 = f_0^{-1} \simeq 6.28\times 10^{-5}\ \text{s} = 62.8\ \mu\text{s}$$

(b)
$$x(t) = 4 + \frac{10}{2} e^{\hat{j}(10^5 t + \frac{2\pi}{3})} + \frac{10}{2} e^{-\hat{j}(10^5 t + \frac{2\pi}{3})} + \frac{14 e^{\hat{j}\pi}}{2\hat{j}} e^{\hat{j}(3\times 10^5 t)} + \frac{14}{2\hat{j}} e^{-\hat{j}(3\times 10^5 t)}$$

$$= \underbrace{4}_{x_0} + \underbrace{5e^{\hat{j}\frac{2\pi}{3}}}_{x_1} e^{\hat{j}10^5 t} + \underbrace{5e^{-\hat{j}\frac{2\pi}{3}}}_{x_{-1}} e^{-\hat{j}10^5 t} + \underbrace{7e^{\hat{j}\pi/2}}_{x_3} e^{\hat{j}3\times 10^5 t} + \underbrace{7e^{-\hat{j}\pi/2}}_{x_{-3}} e^{-\hat{j}3\times 10^5 t}$$



Exercise # 1 Given $x(t) = 8\cos(20\pi t) - 6\sin(40\pi t)$

(a) Find ω_0, f_0, and T_0.

(b) Express $x(t)$ in complex exponential Fourier series form and sketch its two-sided spectrum.

Exercise # 2 Repeat Exercise # 1 for the following signals:

(i) $x(t) = -10 + 10\sin(100\pi t) - 10\cos\left(400\pi t - \frac{\pi}{4}\right)$

(ii) $x(t) = 2 - 8\sin\left(4000\pi t - \frac{\pi}{6}\right) - 6\cos\left(8000\pi t + \frac{\pi}{3}\right)$

(iii) $x(t) = 4 - 8\cos\left(10^6 t - \frac{5\pi}{6}\right) + 2\sin\left(3\times 10^6 t - \frac{3\pi}{4}\right)$