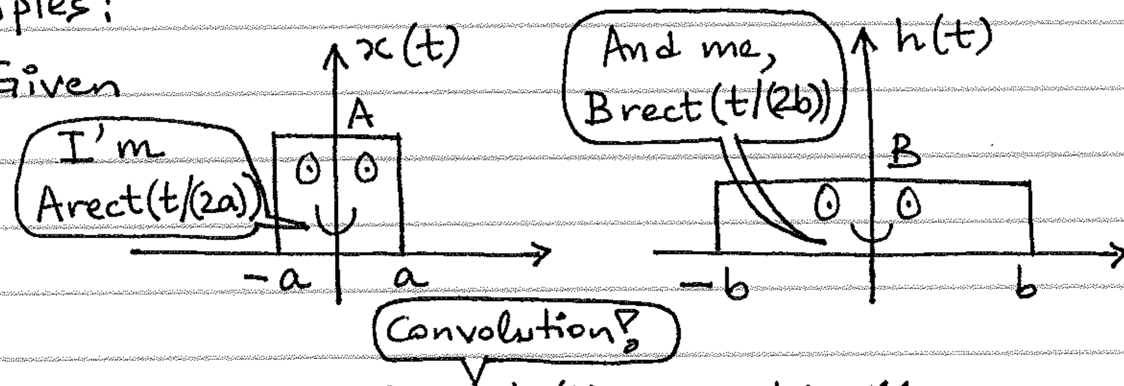
EE 262
SPRING 2013
A. INAN

CONVOLUTION INTEGRAL

Examples:

1. Given

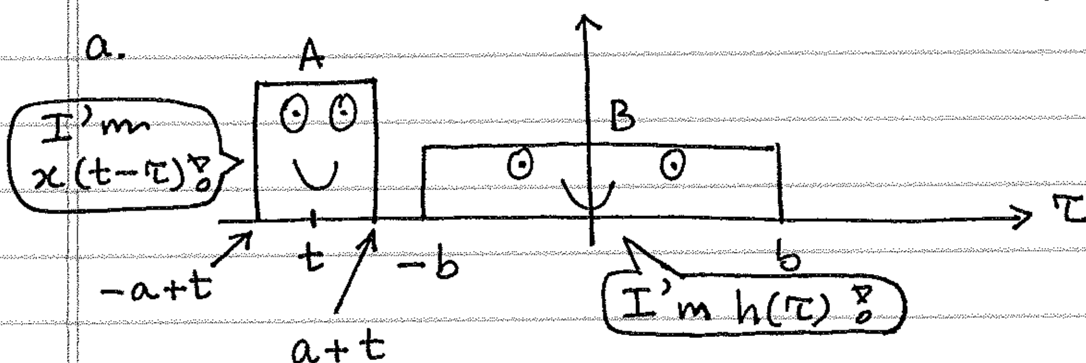


a. Find $y(t) = x(t) * h(t)$ graphically.

b. Find $y(t) = x(t) * h(t)$ using convolution of special signals and properties of convolution.

Solution: For $a+t \leq -b \rightarrow t \leq -(a+b)$

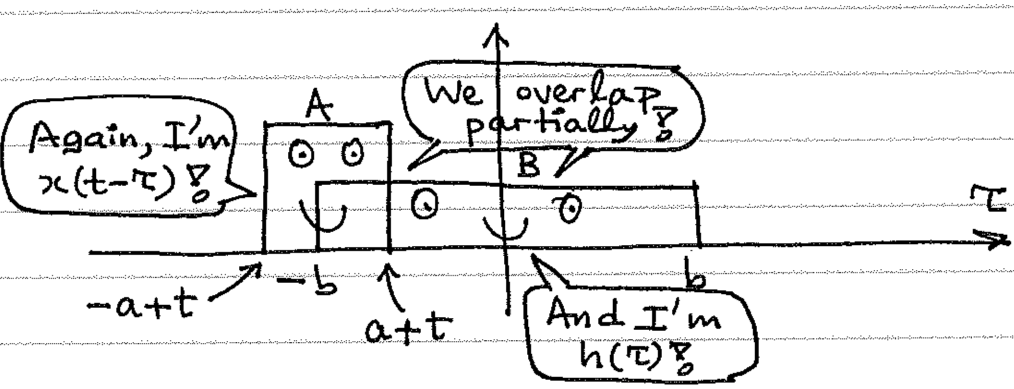
a.



$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau = \boxed{0}$$

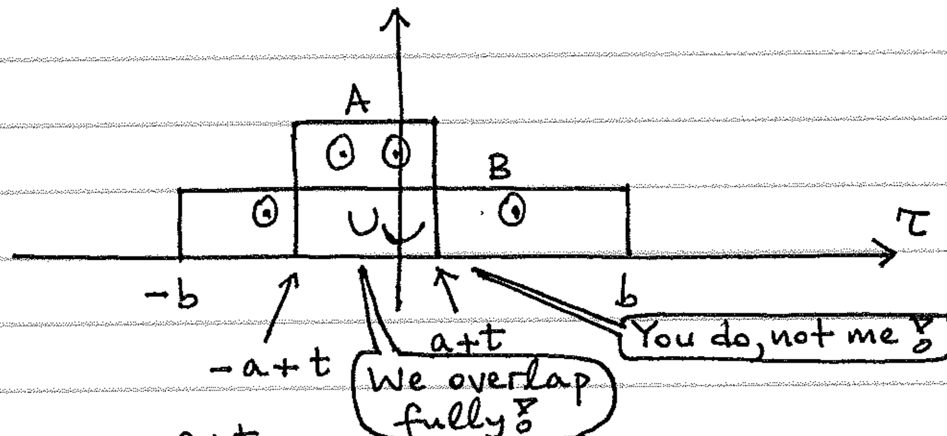
We don't overlap! For this reason, our product is zero!

For $a+t \geq -b$ & $-a+t \leq -b \rightarrow -(a+b) \leq t \leq a-b$



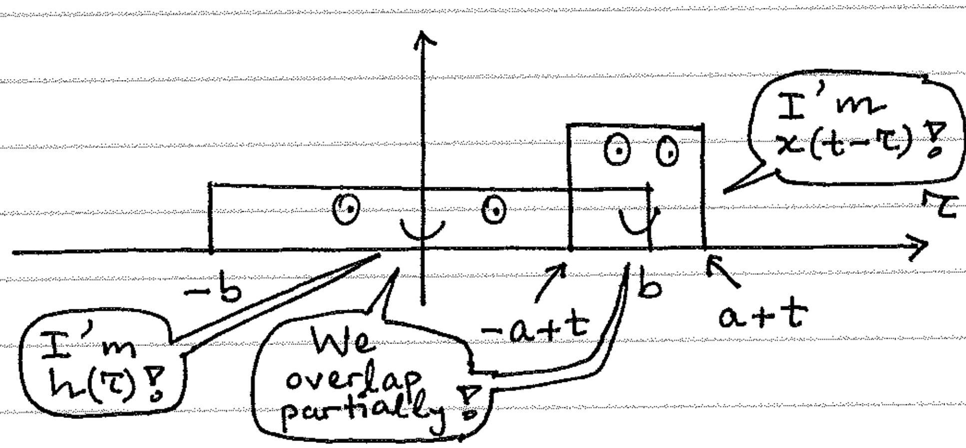
$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau = \int_{-b}^{a+t} (A)(B)\,d\tau = \boxed{AB(t+a+b)}$$

For $a+t \leq b$ & $-a+t \geq -b \rightarrow a-b \leq t \leq b-a$



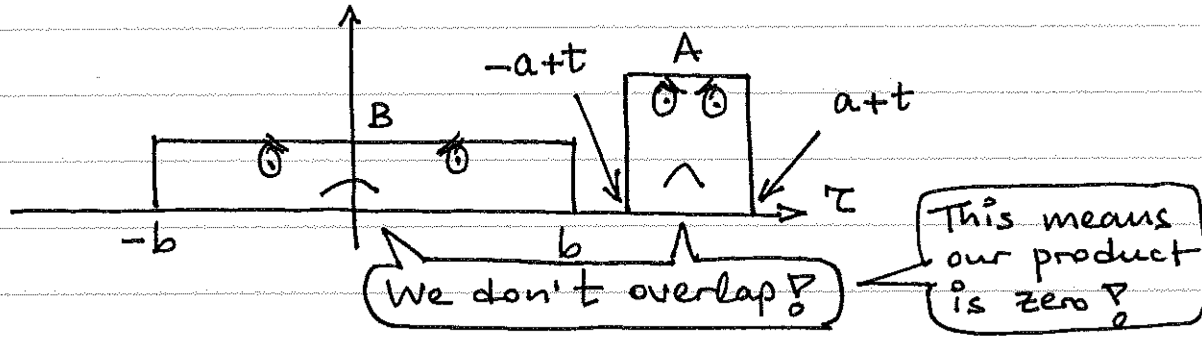
$$y(t) = \int_{-a+t}^{a+t} (A)(B)\,d\tau = AB(\cancel{t}+a-(\cancel{t}-a)) = \boxed{2aAB}$$

For $a+t \geq b$ & $-a+t \leq b \rightarrow b-a \leq t \leq b+a$



$$y(t) = \int_{-a+t}^{b} (A)(B)\,d\tau = AB(b+a-t) = \boxed{-AB(t-a-b)}$$

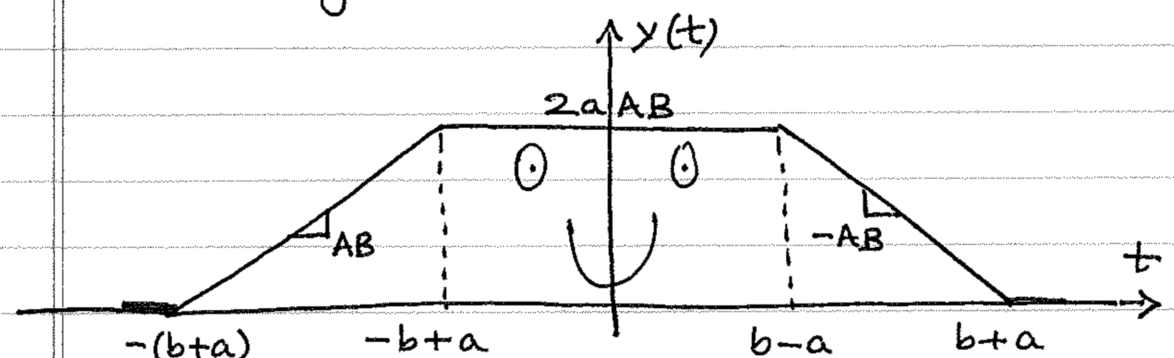
For $-a+t \geq b \rightarrow t \geq a+b$



$$y(t) = \boxed{0}$$

$$\boxed{y(t) = \begin{cases} 0 & , -\infty < t \leq -(b+a) \\ AB(t+a+b) & , -(b+a) \leq t \leq -b+a \\ 2aAB & , -b+a \leq t \leq b-a \\ -AB(t-a-b) & , b-a \leq t \leq b+a \\ 0 & , b+a \leq t < \infty \end{cases}}$$

Sketching $y(t)$ vs. t ;



$y(t)$ can also be expressed in terms of ramp functions as :

$$\boxed{\begin{aligned} y(t) = & ABr(t+b+a) - ABr(t+b-a) \\ & - ABr(t-b+a) + ABr(t-b-a) \end{aligned}}$$

b. Express both $x(t)$ and $h(t)$ in terms of step functions:

$$x(t) = Au(t+a) - Au(t-a)$$

$$h(t) = Bu(t+b) - Bu(t-b)$$

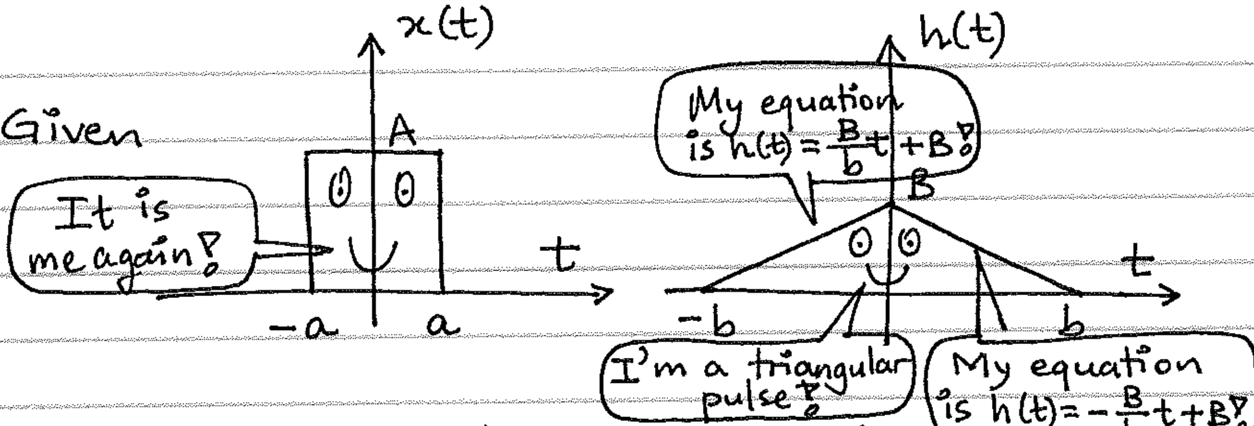
Since $u(t) * u(t) = t\,u(t) = r(t)$, using the distributive and time-shift properties of convolution:

$$\underbrace{x(t) * h(t)}_{y(t)} = \left[Au(t+a) - Au(t-a)\right] * \left[Bu(t+b) - Bu(t-b)\right]$$

$$= Au(t+a) * Bu(t+b)$$
$$+ Au(t+a) * (-Bu(t-b))$$
$$- Au(t-a) * Bu(t+b)$$
$$- Au(t-a) * (-Bu(t-b))$$

$$= \boxed{\begin{aligned} &ABr(t+a+b) - ABr(t+a-b) \\ &- ABr(t-a+b) + ABr(t-a-b) \end{aligned}}$$

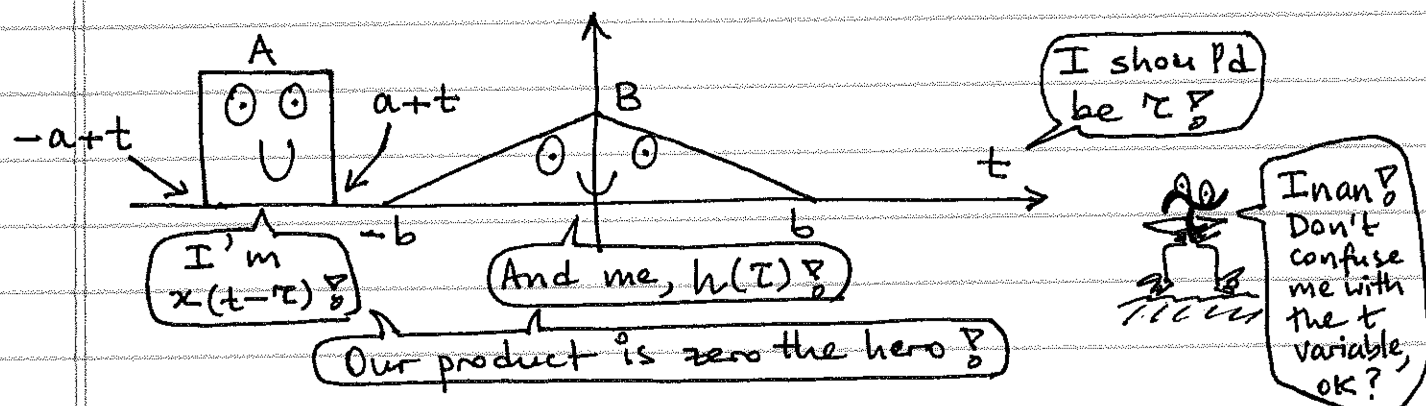
2. Given



a. Find $y(t) = x(t) * h(t)$ graphically.

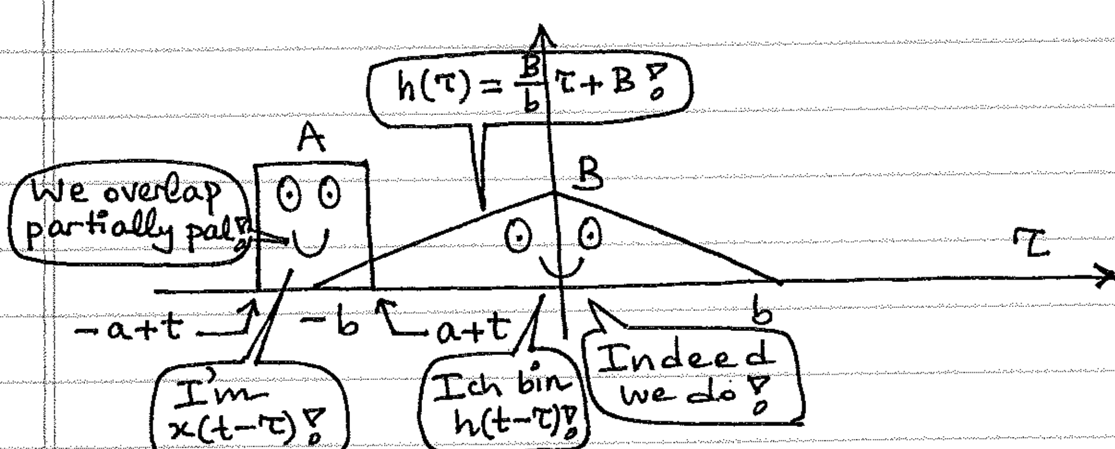
b. Find $y(t) = x(t) * h(t)$ using convolution of special signals and properties of convolution.

Solution: For $a+t \leq -b \rightarrow t \leq -(a+b)$



$$y(t) = \int_{-\infty}^{\infty} \underbrace{x(t-\tau)\,h(\tau)}_{0}\, d\tau = \boxed{0}$$

For $a+t \geq -b$ & $-a+t \leq -b \rightarrow -(a+b) \leq t \leq a-b$



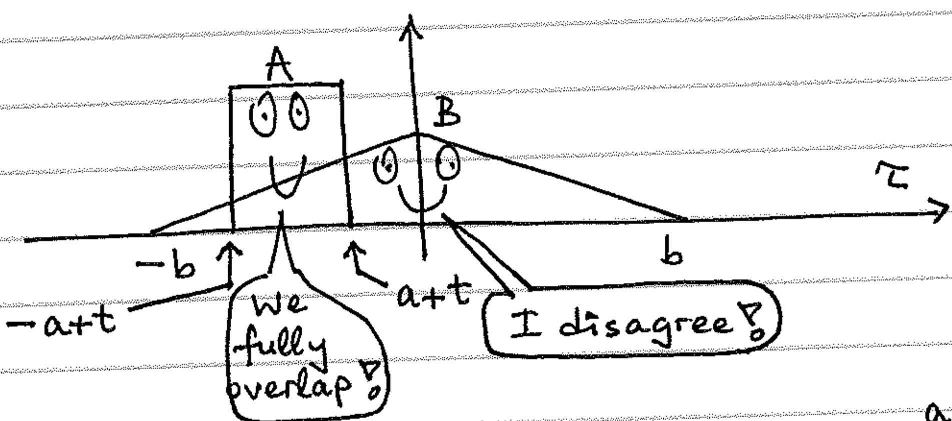
$$y(t) = \int_{-b}^{a+t} \overbrace{A}^{\text{I represent } x(t-\tau)!} \underbrace{\left[\frac{B\tau}{b}+B\right]}_{\text{And I'm } h(\tau)!} d\tau = \left[\frac{AB\tau^2}{2b} + AB\tau\right]_{-b}^{a+t}$$

I'm a parabola!

$$= \boxed{\frac{AB(t+a)^2}{2b} + AB(t+a) + \frac{ABb}{2}}$$

Assuming $2a < b$, for $a+t \leq 0$ & $-a+t \geq -b \overset{\text{or}}{\longleftrightarrow}$

$$a-b \leq t \leq -a$$



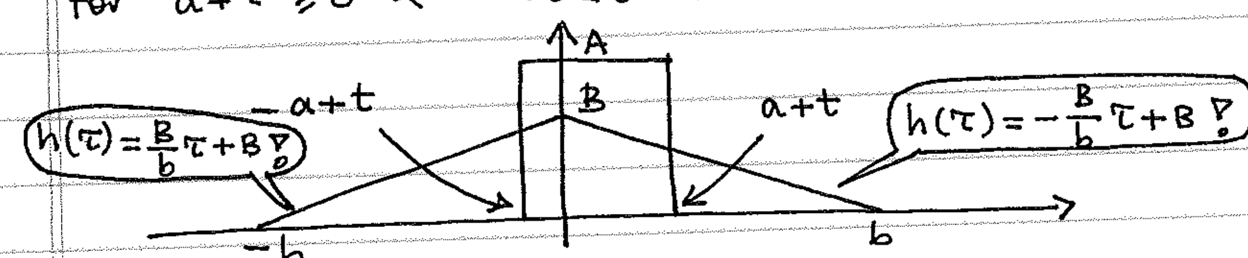
$$y(t) = \int_{-a+t}^{a+t} A\left[\frac{B\tau}{b}+B\right] d\tau = \left[\frac{AB\tau^2}{2b} + AB\tau\right]_{-a+t}^{a+t}$$

$$= \frac{AB(t+a)^2}{2b} + AB(t+a) - \frac{AB(t-a)^2}{2b} - AB(t-a)$$

$$= \boxed{2aAB\left(\frac{t}{b}+1\right)}$$

I'm a straight line!

For $a+t \geq 0$ & $-a+t \leq 0 \longrightarrow \quad -a \leq t \leq a$



$$y(t) = \int_{-a+t}^{0} A\left[\frac{B\tau}{b} + B\right] d\tau + \int_{0}^{a+t} A\left[-\frac{B\tau}{b} + B\right] d\tau$$

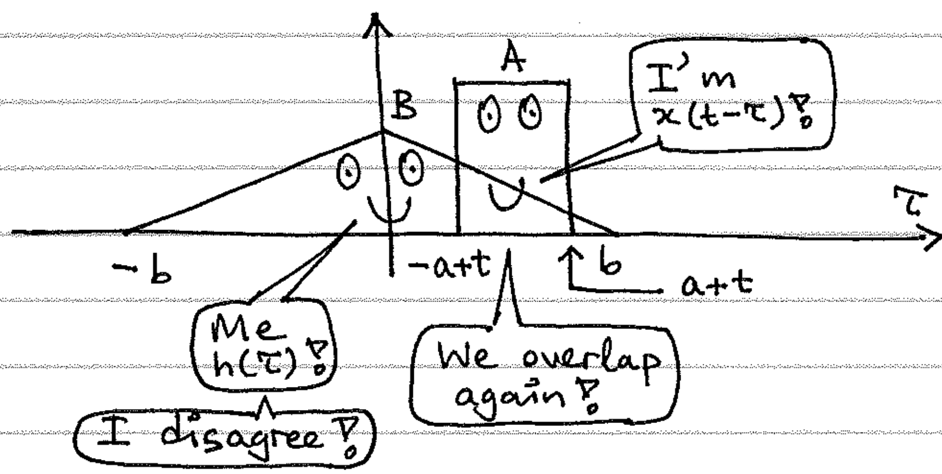
$$= \left[\frac{AB\tau^2}{2b} + AB\tau\right]_{-a+t}^{0} + \left[-\frac{AB\tau^2}{2b} + AB\tau\right]_{0}^{a+t}$$

$$= -\frac{AB(t-a)^2}{2b} - AB(t-a) - \frac{AB(t+a)^2}{2b} + AB(t+a)$$

$$= -\frac{ABt^2}{b} - \frac{ABa^2}{b} + 2aAB = \boxed{-\frac{AB}{b}\left(t^2 + a^2 - 2ab\right)}$$

Parabola!

For $a+t \leq b$ & $-a+t \geq 0 \rightarrow a \leq t \leq b-a$



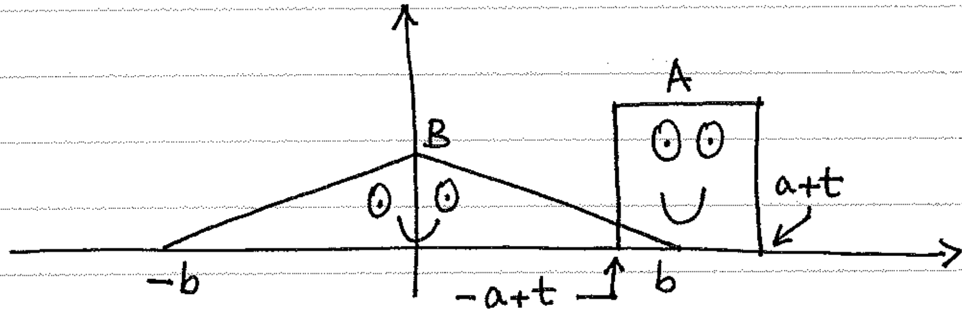
$$y(t) = \int_{-a+t}^{a+t} A\left[-\frac{B\tau}{b} + B\right] d\tau = \left[-\frac{AB\tau^2}{2b} + AB\tau\right]_{-a+t}^{a+t}$$

$$= -\frac{AB(t+a)^2}{2b} + AB(t+a) + \frac{AB(t-a)^2}{2b} - AB(t-a)$$

$$= \boxed{-2aAB\left(\frac{t}{b} - 1\right)}$$

I'm a straight line!

For $a+t \geq b$ & $-a+t \leq b \rightarrow \quad b-a \leq t \leq b+a$



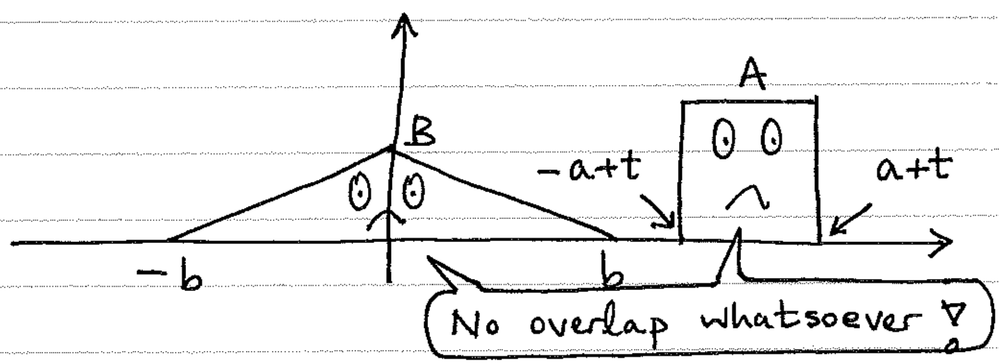
$$y(t) = \int_{-a+t}^{b} A\left[-\frac{B\tau}{b} + B\right] d\tau = \left[-\frac{AB\tau^2}{2b} + AB\tau\right]_{-a+t}^{b}$$

$$= -\frac{ABb}{2} + ABb + \frac{AB(t-a)^2}{2b} - AB(t-a)$$

$$= \boxed{\frac{AB(t-a)^2}{2b} - AB(t-a) + \frac{ABb}{2}}$$

I'm a parabola!

For $-a+t \geq b \rightarrow t \geq b+a$



$$y(t) = \boxed{0}$$

$$y(t) = \begin{cases} 0 & , \quad t \le -(a+b) \\ \dfrac{AB(t+a)^2}{2b} + AB(t+a) + \dfrac{ABb}{2} & , -(a+b) \le t \le a-b \\ 2aAB\left(\dfrac{t}{b}+1\right) & , \quad a-b \le t \le -a \\ -\dfrac{AB}{b}(t^2+a^2-2ab) & , \quad -a \le t \le a \\ -2aAB\left(\dfrac{t}{b}-1\right) & , \quad a \le t \le b-a \\ \dfrac{AB(t-a)^2}{2b} - AB(t-a) + \dfrac{ABb}{2} & , \quad b-a \le t \le b+a \\ 0 & , \quad t \ge b+a \end{cases}$$

b. Note that $x(t) = Au(t+a) - Au(t-a)$

$$h(t) = \frac{B}{b}r(t+b) - \frac{2B}{b}r(t) + \frac{B}{b}r(t-b)$$

Also note that $u(t) * r(t) = \frac{t^2}{2}u(t)$

Using the distributive and time-shift properties of convolution along with $u(t) * r(t) = \frac{t^2}{2}u(t)$, we obtain:

$$y(t) = x(t) * h(t)$$

$$= \left[Au(t+a) - Au(t-a)\right] * \left[\frac{B}{b} r(t+b) - \frac{2B}{b} r(t) + \frac{B}{b} r(t-b)\right]$$

$$= \frac{AB}{2b}(t+a+b)^2 u(t+a+b) - \frac{AB}{2b}(t-a+b)^2 u(t-a+b)$$

$$- \frac{AB}{b}(t+a)^2 u(t+a) + \frac{AB}{b}(t-a)^2 u(t-a)$$

$$+ \frac{AB}{2b}(t+a-b)^2 u(t+a-b) - \frac{AB}{2b}(t-a-b)^2 u(t-a-b)$$