University ☺f P☺rtland
Sch☺☺l ☺f Engineering

EE 301
Spring 2014
A.Inan

Solutions to Homework # 4
(Wednesday, March 19, 2014)

(1) Electrical length of transmission lines. Find the electrical lengths of the following transmission lines:

(a) 100-kilometer long air transmission line at 60 Hz;
(b) 100-meter long coaxial line with a velocity factor of 0.67 at 300 MHz.

Solution:

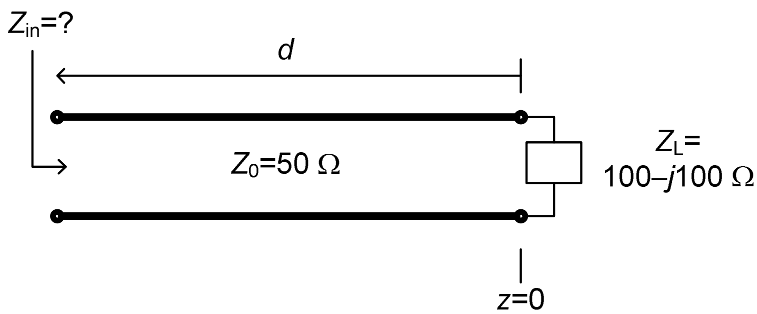
(a) Since air line, $v_p = c \cong 3\times10^5 \text{ km-s}^{-1}$. So, the electrical length of the 100-kilometer air line at 60 Hz is given by

$$\bar{d} = \frac{d}{\lambda} = \frac{d}{(v_p / f)} \cong \frac{(100\text{ km})}{(3\times10^5 \text{ km-s}^{-1})/(60\text{ Hz})} \cong 0.02$$

(b) The electrical length of the 100-meter coaxial line with velocity factor 0.67 at 300 MHz can be calculated as

$$\bar{d} = \frac{d}{\lambda} = \frac{d}{(v_p / f)} \cong \frac{(100\text{ m})}{(0.67\times3\times10^8 \text{ m-s}^{-1})/(3\times10^8 \text{ Hz})} \cong 149.25$$

(2) Load reflection coefficient, standing wave ratio, and input impedance. A 50 Ω transmission line is terminated with a capacitive load having a load impedance of $Z_L = 100 - j100\,\Omega$, as shown. (a) Find the load reflection coefficient, i.e., Γ_L. (b) Find the standing wave ratio S on the line. (c) Find the input impedance of the line, Z_{in}, for four different line lengths: $d_1 = 0.125\lambda$, $d_2 = 0.25\lambda$, $d_3 = 0.375\lambda$, and $d_4 = 0.5\lambda$, respectively.



Solution:

(a) The load reflection coefficient Γ_L is given by

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 - j100 - 50}{100 - j100 + 50} = \frac{50 - j100}{150 - j100} = \frac{1 - j2}{3 - j2} \times \underbrace{\frac{3 + j2}{3 + j2}}_{\substack{\text{complex}\\ \text{conjugate}\\ \text{multiplication!}}} = \frac{7 - j4}{13} \cong \underbrace{0.620}_{|\Gamma_L|} e^{j\left(\underbrace{-29.7^\circ}_{\phi_L}\right)}$$

(b) The standing wave ratio S on the line can be found as

$$S = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \cong \frac{1 + 0.620}{1 - 0.620} \cong 4.27$$

(c) For $d_1 = 0.125\lambda$, the input impedance Z_{in1} of the line is given by

$$Z_{in1} = Z_0 \frac{Z_L + jZ_0 \tan(2\pi d_1/\lambda)}{Z_0 + jZ_L \tan((2\pi d_1/\lambda))} = 50 \frac{(100 - j100) + j50 \tan\left(\underbrace{2\pi \times 0.125}_{\pi/4}\right)}{50 + j(100 - j100)\tan\left(\underbrace{2\pi \times 0.125}_{\pi/4}\right)}$$

$$= 50 \frac{100 - j100 + j50}{50 + j(100 - j100)} = 50 \frac{100 - j50}{150 + j100} = 50 \frac{2 - j}{3 + j2} \times \underbrace{\frac{3 - j2}{3 - j2}}_{\substack{\text{complex}\\ \text{conjugate}\\ \text{multiplication!}}}$$

$$= 50 \frac{4 - j7}{13} \cong 15.38 - j26.92\,\Omega$$

For $d_2 = 0.25\lambda$ (i.e., quarter wavelength), the input impedance Z_{in2} is given by

$$Z_{in2} = \frac{{Z_0}^2}{Z_L} = \frac{50^2}{(100 - j100)} \times \underbrace{\frac{100 + j100}{100 + j100}}_{\substack{\text{complex}\\ \text{conjugate}\\ \text{multiplication!}}} = \frac{50^2(100 + j100)}{\underbrace{100^2 + 100^2}_{2\times 10^4}} = 12.5 + j12.5\,\Omega$$

For $d_3 = 0.375\lambda$, the input impedance Z_{in3} is given by

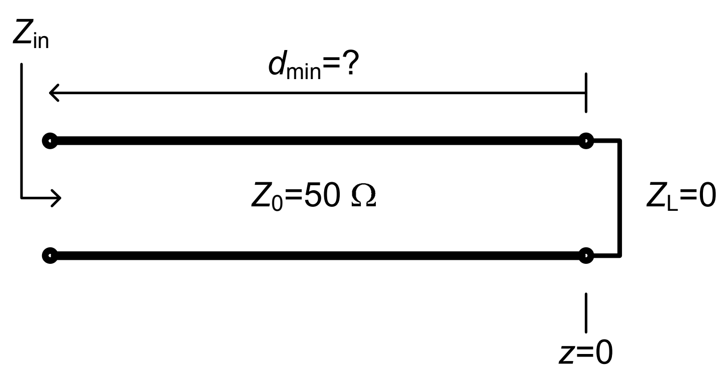
$$Z_{\text{in3}} = Z_0 \frac{Z_L + jZ_0 \tan(2\pi d_3/\lambda)}{Z_0 + jZ_L \tan((2\pi d_3/\lambda))} = 50 \frac{(100 - j100) + j50 \tan\left(\underbrace{2\pi \times 0.375}_{3\pi/4}\right)}{50 + j(100 - j100) \tan\left(\underbrace{2\pi \times 0.375}_{3\pi/4}\right)}$$

$$= 50 \frac{100 - j100 - j50}{50 - j(100 - j100)} = 50 \frac{100 - j150}{-50 - j100} = 50 \frac{2 - j3}{-1 - j2} \times \underbrace{\frac{-1 + j2}{-1 + j2}}_{\substack{\text{complex} \\ \text{conjugate} \\ \text{multiplication!}}}$$

$$= 50 \frac{4 + j7}{5} = 40 + j70\,\Omega$$

For $d_4 = 0.5\lambda$ (i.e., half wavelength), the input impedance Z_{in4} is given by $Z_{\text{in4}} = Z_L = 100 - j100\,\Omega$.

(3) Input impedance of a transmission line. Consider a short-circuited 50 Ω transmission line (a short-circuited stub) as shown. Find the shortest electrical length of the line such that (a) $Z_{\text{in}} = j50\,\Omega$; (b) $Z_{\text{in}} = -j150\,\Omega$; (c) $Z_{\text{in}} = \infty$, and (d) $Z_{\text{in}} = 0$.



Solution:

(a) The input impedance of a short-circuited transmission line is given by

$$Z_{\text{in}_{\text{S.C.}}} = jZ_0 \tan\left(\frac{2\pi}{\lambda} d\right)$$

So, substituting the values $Z_{\text{in}} = j50$ Ω and $Z_0 = 50$ Ω, we have

$$Z_{\text{in}_{\text{S.C.}}} = j50 \tan\left(\frac{2\pi}{\lambda} d\right) = j50\,\Omega \to \bar{d} = \frac{d}{\lambda} = \frac{1}{2\pi} \underbrace{\tan^{-1} 1}_{\pi/4 \text{ rad}} = 0.125$$

(b) Substituting the values $Z_{\text{in}} = -j150$ Ω and $Z_0 = 50$ Ω, we have

$$Z_{\text{in}_{\text{S.C.}}} = j50\tan\left(\frac{2\pi}{\lambda}d\right) = -j150\,\Omega \rightarrow \bar{d} = \frac{d}{\lambda} = \frac{1}{2\pi}\underbrace{\tan^{-1}(-3)}_{\sim -1.249\text{ rad}} \cong -0.1988 + 0.5 = 0.3012$$

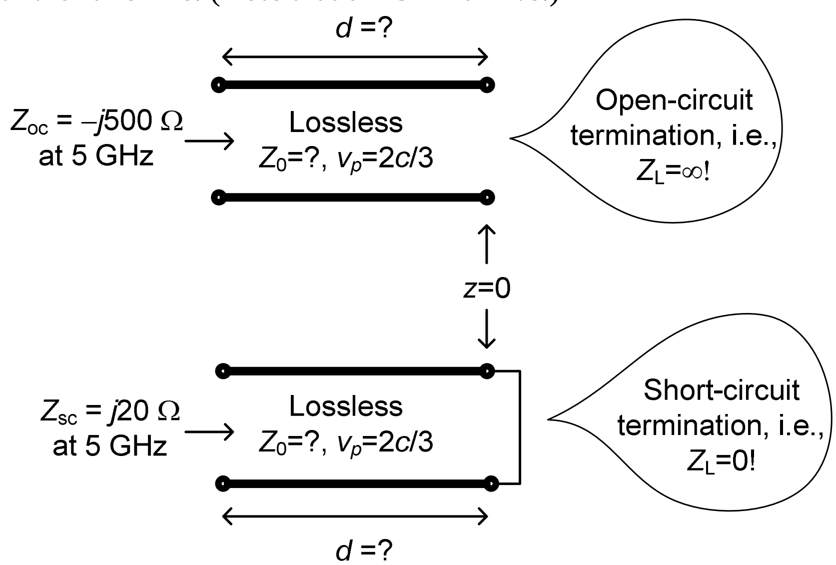
(c) Substituting the values $Z_{\text{in}} = \infty$ and $Z_0 = 50\ \Omega$, we have

$$Z_{\text{in}_{\text{S.C.}}} = j50\tan\left(\frac{2\pi}{\lambda}d\right) = \infty \rightarrow \bar{d} = \frac{d}{\lambda} = \frac{1}{2\pi}\underbrace{\tan^{-1}(\infty)}_{(\pi/2)\text{ rad}} = \frac{1}{2\pi}\times\frac{\pi}{2} = 0.25$$

(d) Substituting the values $Z_{\text{in}} = 0$ and $Z_0 = 50\ \Omega$, we have

$$Z_{\text{in}_{\text{S.C.}}} = j50\tan\left(\frac{2\pi}{\lambda}d\right) = 0 \rightarrow \bar{d} = \frac{d}{\lambda} = \frac{1}{2\pi}\underbrace{\tan^{-1}(0)}_{\pi\text{ rad}} = \frac{1}{2\pi}\times\pi = 0.5$$

(4) Open- and short-circuit impedance measurements. If the open- and short-circuit terminated input impedances of a lossless transmission line with characteristic impedance Z_0, length d and phase velocity $v_p = 2c/3$ are measured at 5 GHz to be $Z_{oc} = -j500\ \Omega$ and $Z_{sc} = j20\ \Omega$ respectively, calculate Z_0 and the length d of this line. (Note that $c = 3 \times 10^8$ m/s.)



<u>Solution:</u>

The input impedances of the open-circuited and short-circuited lines are given by

$$Z_{\text{in}_{\text{O.C.}}} = \frac{-jZ_0}{\tan\left(\frac{2\pi}{\lambda}d\right)} = -j500\,\Omega \quad \text{and} \quad Z_{\text{in}_{\text{S.C.}}} = jZ_0\tan\left(\frac{2\pi}{\lambda}d\right) = j20\,\Omega$$

So, by multiplying the above two equations yields

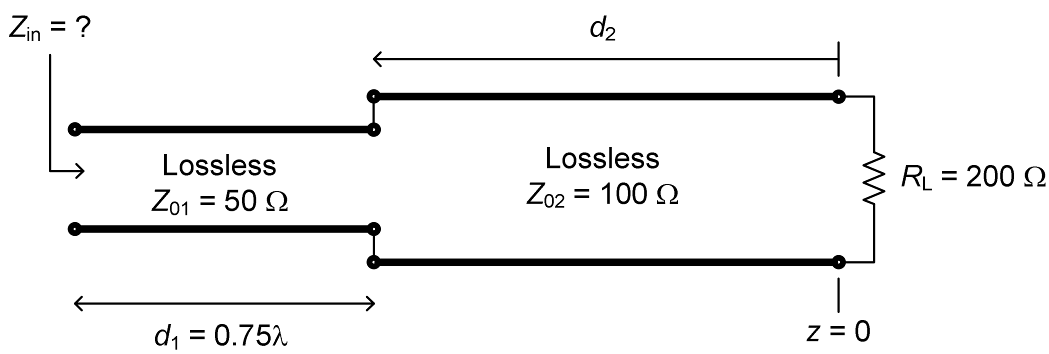
$$Z_{\mathrm{in}_{\mathrm{S.C.}}} \times Z_{\mathrm{in}_{\mathrm{O.C.}}} = jZ_0 \tan\left(\frac{2\pi}{\lambda}d\right) \times \frac{-jZ_0}{\tan(2\pi d/\lambda)} = Z_0^2 = (j20)(-j500) = 10^4\ \Omega^2 \rightarrow Z_0 = 100\,\Omega$$

The electrical length of the line can then be obtained by substituting the Z_0 value calculated in one of the above equations as

$$Z_{\mathrm{in}_{\mathrm{S.C.}}} = j100 \tan\left(\frac{2\pi}{\lambda}d\right) = j20\,\Omega \rightarrow \bar{d} = \frac{d}{\lambda} = \frac{1}{2\pi}\underbrace{\tan^{-1}(0.2)}_{\sim 0.1974\ \mathrm{rad}} \cong 0.0314$$

Since $f = 5$ GHz and $v_p = 2c/3$, $\lambda = v_p / f = 2 \times 10^8$ m-s^{-1} / 5×10^9 Hz = 0.04 m = 4 cm. Therefore, the actual length of the line can be found from the electrical length as d = (0.0314)(4 cm) = 0.126 cm.

(5) Input impedance. Find the input impedance Z_{in} of the transmission line system shown for (a) d_2=0.25λ; (b) d_2=0.5λ; and (c) d_2=0.125λ.



Solution:

(a) For d_2=0.25λ, the input impedance of the second line can be calculated as

$$Z_{\mathrm{in2}} = \frac{Z_{02}^2}{Z_{\mathrm{L}}} = \frac{(100)^2}{200} = 50\,\Omega$$

The input impedance of the first line is then given by

$$Z_{\mathrm{in1}} = \frac{Z_{01}^2}{Z_{\mathrm{in2}}} = \frac{(50)^2}{50} = 50\,\Omega$$

(b) For d_2=0.5λ, the input impedance of the second line can be calculated as

$$Z_{\mathrm{in2}} = Z_{\mathrm{L}} = 200\,\Omega$$

The input impedance of the first line is then given by

$$Z_{\text{in1}} = \frac{Z_{01}^2}{Z_{\text{in2}}} = \frac{(50)^2}{200} = 12.5\,\Omega$$

(c) For d_2=0.125λ, the input impedance of the second line can be calculated as

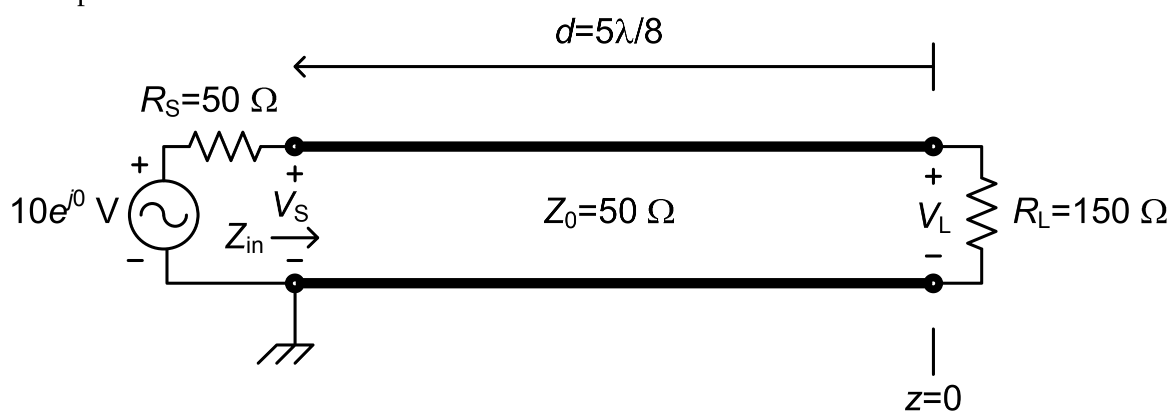
$$Z_{\text{in2}} = Z_{02}\frac{Z_{\text{L}} + jZ_{02}\tan(2\pi d_2/\lambda)}{Z_{02} + jZ_{\text{L}}\tan((2\pi d_2/\lambda))} = 100\frac{200 + j100\tan\left(\underbrace{2\pi\times 0.125}_{\pi/4}\right)}{100 + j200\tan\left(\underbrace{2\pi\times 0.125}_{\pi/4}\right)}$$

$$= 100\frac{200 + j100}{100 + j200} = 100\frac{2+j}{1+j2}\times\underbrace{\frac{1-j2}{1-j2}}_{\substack{\text{complex}\\\text{conjugate}\\\text{multiplication!}}} = 100\frac{2+2+j-j4}{5}\,\Omega = 80 - j60\,\Omega$$

The input impedance of the first line is then given by

$$Z_{\text{in1}} = \frac{Z_{01}^2}{Z_{\text{in2}}} = \frac{(50)^2}{80 - j60} = \frac{125}{4 - j3}\times\underbrace{\frac{4+j3}{4+j3}}_{\substack{\text{complex}\\\text{conjugate}\\\text{multiplication!}}} = \frac{125(4+j3)}{25} = 20 + j15\,\Omega$$

(6) Source- and load-end voltages. For the transmission line shown, assuming sinusoidal steady state, calculate the source-end and load-end voltages V_S and V_L in phasor form.



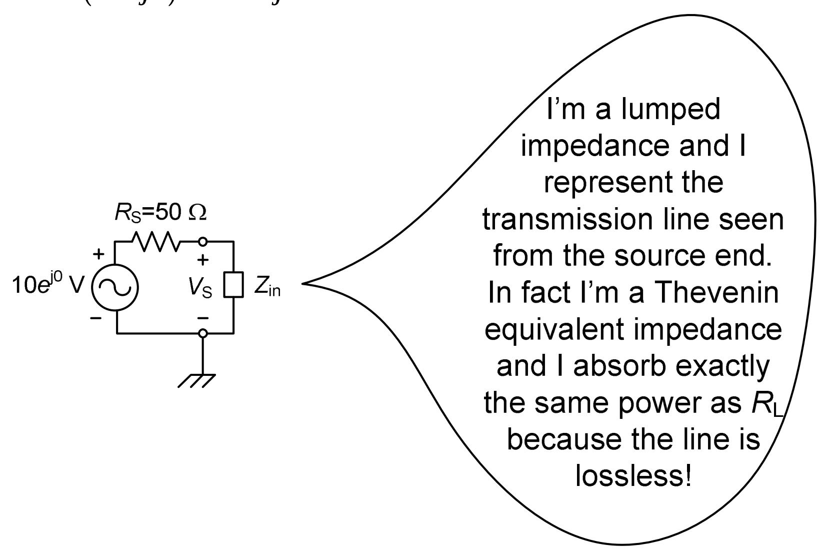
Solution:

The input impedance of a uniform lossless transmission line is given by

$$Z_{\text{in}} = Z_0 \frac{Z_L + jZ_0 \tan(2\pi d/\lambda)}{Z_0 + jZ_L \tan((2\pi d/\lambda))} = 50 \frac{150 + j50 \tan\left(\underbrace{2\pi \times 0.625}_{5\pi/4}\right)}{50 + j150 \tan\left(\underbrace{2\pi \times 0.625}_{5\pi/4}\right)}$$

$$= 50 \frac{150 + j50}{50 + j150} = 50 \frac{3 + j}{1 + j3} \times \underbrace{\frac{1 - j3}{1 - j3}}_{\substack{\text{complex} \\ \text{conjugate} \\ \text{multiplication!}}} = 50 \frac{3 + 3 + j - j9}{1 + 9}$$

$$= 5(6 - j8) = 30 - j40\,\Omega$$



Using the voltage divider principle, the source-end phasor voltage V_S is obtained as

$$V_S = \frac{Z_{\text{in}}}{R_S + Z_{\text{in}}}\left(10e^{j0}\right) = \frac{30 - j40}{\underbrace{50 + 30 - j40}_{80 - j40}}\left(10e^{j0}\right) \cong \frac{50e^{-j53.13^\circ}}{89.44e^{-j26.57^\circ}}\left(10e^{j0}\right) \cong 5.59e^{-j26.57^\circ}\ \Omega$$

Note that the total phasor voltage at any position along the transmission line is given by

$$V(z) = \underbrace{V^+e^{-j\beta z}}_{\substack{\text{Incident} \\ \text{voltage} \\ \text{wave}}} + \underbrace{V^-e^{+j\beta z}}_{\substack{\text{Reflected} \\ \text{voltage} \\ \text{wave}}} = V^+e^{-j\beta z}\left[1 + \Gamma_L e^{+j2\beta z}\right]$$

Note that the load reflection coefficient is given by

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{150 - 50}{150 + 50} = 0.5$$

Using the values of Γ_L and V_S, we can obtain the value of V^+ as follows:

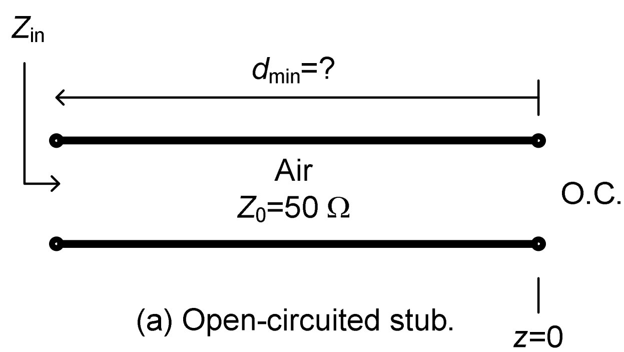
$$\underbrace{V\left(z=-\frac{5\lambda}{8}\right)}_{V_S}=V^+e^{j5\pi/4}\left[1+\Gamma_L\underbrace{e^{-j5\pi/2}}_{-j}\right]=V^+e^{j5\pi/4}\left[1-j0.5\right]\cong 5.59e^{-j26.57^\circ}$$

$$\rightarrow V^+\cong 5e^{-j225^\circ}\text{ V}=5e^{j135^\circ}\text{ V}$$

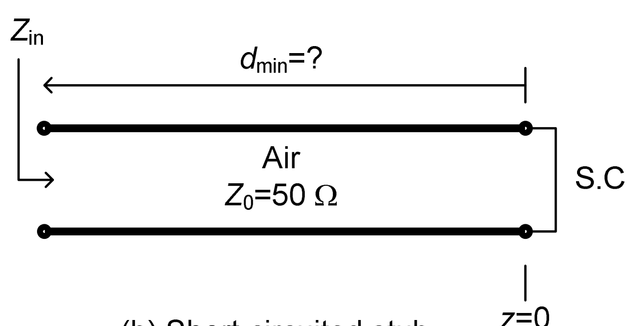
Next, the load-end phasor voltage can be calculated as

$$V_L=V(z=0)=V^+\left[1+\underbrace{\Gamma_L}_{0.5}\right]\cong 5e^{j135^\circ}\times 1.5=7.5e^{j135^\circ}\text{ V}$$

(7) Designing a capacitor using a stub. Capacitive and inductive circuit elements can be designed using short-circuit or open-circuited stubs. The lengths of these stubs are typically short with respect to the associated wavelength. (a) Design an open-circuited 50 Ω air stub that will provide the impedance of a 4 nF capacitor at 10 GHz. Find the shortest length of the stub. (b) Redesign the capacitor in part (a) using a short-circuited 50 Ω air stub. (c) Which design yields the shortest length and why?



(a) Open-circuited stub.



(b) Short-circuited stub.

Solution:

(a) Note that $\lambda_{air}\cong\left(3\times10^{10}\text{ cm-s}^{-1}\right)/\left(10^{10}\text{ Hz}\right)=3\,\text{cm}$. Equating the input impedance expression of an open-circuited stub to the impedance of a lumped capacitor, we have

$$Z_{\text{in}_{\text{O.C.}}} = \frac{-jZ_0}{\tan\left(\frac{2\pi}{\lambda}d\right)} = Z_{\text{cap}} = \frac{-j}{\omega C} \rightarrow \frac{-j50}{\tan\left(\frac{2\pi}{\lambda}d\right)} = \frac{-j}{2\pi\times 10^{10}\times \underbrace{4\times 10^{-9}}_{4\text{ nF}}} = \frac{-j}{80\pi}\Omega$$

$$\rightarrow \frac{d_{\min}}{\lambda} = \frac{1}{2\pi}\underbrace{\tan^{-1}(4000\pi)}_{\sim\pi/2\text{ rad}} \cong 0.249987 \rightarrow d_{\min} \cong 0.249987\lambda_{\text{air}} \cong 0.74996\,\text{cm}$$

(b) Using the input impedance expression of a short-circuited stub, we have

$$Z_{\text{in}_{\text{S.C.}}} = jZ_0 \tan\left(\frac{2\pi}{\lambda}d\right) = Z_{\text{cap}} = \frac{-j}{\omega C} \rightarrow j50\tan\left(\frac{2\pi}{\lambda}d\right) = \frac{-j}{80\pi}\Omega$$

$$\rightarrow \frac{d_{\min}}{\lambda} = \frac{1}{2\pi}\underbrace{\tan^{-1}\left(\frac{-1}{4000\pi}\right)}_{\sim -0.000079577\text{ rad}} \cong -1.2665\times 10^{-5} + \underbrace{0.5}_{\substack{\text{adjustment}\\ \text{to obtain a}\\ \text{positive } d!}} = 0.499987$$

$$\rightarrow d_{\min} \cong 0.499987\lambda_{\text{air}} \cong 1.49996\,\text{cm}$$

(c) As expected, the open-circuited stub yields the shortest length since the input impedance of the open-circuited stub is capacitive when its electrical length falls in the range $0 < d/\lambda < 0.25$ whereas the input impedance of the short-circuited stub in the same range is inductive.

(8) Designing an inductor using a stub. (a) Design an open-circuited 50 Ω micro-strip transmission-line stub having an effective relative dielectric constant of $\varepsilon_r \cong 6$ that will provide the impedance of a 5 nH inductor at 5 GHz. Find the shortest length of the stub. (b) Repeat the same design using a short-circuited 50 Ω micro-strip line stub having an effective relative dielectric constant of $\varepsilon_r \cong 6$. (c) Which design resulted in a shorter stub and <u>why</u>?

<u>Solution:</u>

(a) Note that $\lambda \cong \left(3\times 10^{10}\ \text{cm-s}^{-1}/\sqrt{6}\right)/\left(5\times 10^{9}\ \text{Hz}\right) \cong 2.45\,\text{cm}$. Equating the input impedance expression of an open-circuited stub to the impedance of a lumped inductor, we have

$$Z_{\text{in}_{\text{O.C.}}} = \frac{-jZ_0}{\tan\left(\frac{2\pi}{\lambda}d\right)} = \underbrace{j\omega L}_{Z_{\text{coil}}} \rightarrow \frac{-j50}{\tan\left(\frac{2\pi}{\lambda}d\right)} = \underbrace{j2\pi\times 5\times 10^{9}\times 5\times 10^{-9}}_{j50\pi}$$

$$\rightarrow \frac{d_{\min}}{\lambda} = \frac{1}{2\pi}\underbrace{\tan^{-1}\left(-\frac{1}{2\pi}\right)}_{\sim -0.1578\text{ rad}} \cong -0.02512 + 0.5 = 0.4749 \rightarrow d_{\min} \cong 0.4749\lambda \cong 1.163\,\text{cm}$$

(d) Using the input impedance expression of a short-circuited stub, we have

$$Z_{\text{in}_{\text{S.C.}}} = jZ_0 \tan\left(\frac{2\pi}{\lambda}d\right) = \underbrace{j\omega L}_{Z_{\text{coil}}} \rightarrow j50\tan\left(\frac{2\pi}{\lambda}d\right) = j50\pi$$

$$\rightarrow \frac{d_{\text{min}}}{\lambda} = \frac{1}{2\pi}\underbrace{\tan^{-1}\pi}_{\sim 1.2626 \text{ rad}} \cong 0.20095 \rightarrow d_{\text{min}} \cong 0.20095\lambda \cong 0.4922 \text{ cm}$$

(e) As expected, the short-circuited stub yields the shortest length since the input impedance of the short-circuited stub is inductive when its electrical length falls in the range $0 < d/\lambda < 0.25$ whereas the input impedance of the open-circuited stub in the same range is capacitive.