UNIVERSITY ☺F P☺RTLAND
Sch☺☺l ☺f Engineering

<u>Basic Mathematical Equations/Identities that Inan's Students Need!</u>

(Prepared by A. S. Inan)

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<u>Basic Trigonometry Table</u>

$p/4 \text{ radians} = 45^\circ$, $p/2 \text{ radians} = 90^\circ$, $p \text{ radians} = 180^\circ$, $3p/2 \text{ radians} = 270^\circ$

q	$\sin q$	$\cos q$	$\tan q = \sin q/\cos q$
0	0	1	0
30° or $p/6$ radians	0.5	$\sqrt{3}/2$	$1/\sqrt{3}$
45° or $p/4$ radians	$1/\sqrt{2}$	$1/\sqrt{2}$	1
60° or $p/3$ radians	$\sqrt{3}/2$	0.5	$\sqrt{3}$
90° or $p/2$ radians	1	0	∞
$90^\circ \le q \le 180^\circ$	$\sin(180^\circ - q)$	$-\cos(180^\circ - q)$	$-\tan(180^\circ - q)$
$180^\circ \le q \le 270^\circ$	$-\sin(q - 180^\circ)$	$-\cos(q - 180^\circ)$	$\tan(q - 180^\circ)$
$270^\circ \le q \le 360^\circ$	$-\sin(360^\circ - q)$	$\cos(360^\circ - q)$	$-\tan(360^\circ - q)$

<u>Some Handy Trigonometric Identities</u>

$$\sin(-q) = -\sin q \quad \& \quad \cos(-q) = \cos q \quad \& \quad \tan(-q) = -\tan q$$

$$\sin(p - q) = \sin q \quad \& \quad \cos(p - q) = -\cos q \quad \& \quad \tan(p - q) = -\tan q$$

$$\sin(p + q) = -\sin q \quad \& \quad \cos(p + q) = -\cos q \quad \& \quad \tan(p + q) = \tan q$$

$$\sin(2q) = 2\sin q \cos q \quad \& \quad \cos(2q) = \cos^2 q - \sin^2 q \quad \& \quad \tan(2q) = \frac{2\tan q}{1 - \tan^2 q}$$

$$\sin^2 q + \cos^2 q = 1$$

$$\cos^2 q = \frac{1 + \cos(2q)}{2} \quad \& \quad \sin^2 q = \frac{1 - \cos(2q)}{2}$$

$$\cos^3 q = \frac{3\cos q + \cos(3q)}{4} \quad \& \quad \sin^3 q = \frac{3\sin q - \sin(3q)}{4}$$

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \quad \& \quad \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\sin a \sin b = \frac{\cos(a - b) - \cos(a + b)}{2} \quad \& \quad \cos a \cos b = \frac{\cos(a + b) + \cos(a - b)}{2}$$

$$\sin a \cos b = \frac{\sin(a + b) + \sin(a - b)}{2}$$

$$A\sin a + B\cos a = \sqrt{A^2 + B^2}\,\cos\left(a + \tan^{-1}(-A/B)\right)$$

Basics of Complex Numbers

$$Z = \overbrace{\underbrace{X}_{\mathrm{Re}\{Z\}} + j\underbrace{Y}_{\mathrm{Im}\{Z\}}}^{\text{Rectangular form}} = \overbrace{\underbrace{|Z|}_{\text{Magnitude}}\; e^{j\underbrace{\phi_z}_{\text{Phase}}}}^{\text{Polar form}}$$

Rectangular to Polar Conversion Formulas

$$|Z| = \sqrt{X^2 + Y^2} \geq 0,\ \phi_z = \tan^{-1}\left(\frac{Y}{X}\right) \leftarrow \text{Caution when using a calculator!}$$

- Calculator will typically provide the correct value of angle θ obtained from $\theta = \tan^{-1}(\bullet)$ only if θ is in the 1st or the 4th quadrant (i.e., $-90^\circ \leq \theta \leq 90^\circ$)
- Calculator will typically provide the value of angle $\theta = \tan^{-1}(\bullet)$ in degrees, so, if you need the value of θ in radians, you have to multiply the value of θ in degrees by $\pi/180 \rightarrow \theta$ (radians) $= \theta$ (degrees)$\times \pi/180$

Polar to Rectangular Conversion Formulas

$$X = \mathrm{Re}\{Z\} = |Z|\cos\phi_z,\ Y = \mathrm{Im}\{Z\} = |Z|\sin\phi_z$$

Addition, Subtraction, Multiplication, & Division with Complex Numbers

$$Z_1 = X_1 + Y_1 = |Z_1|e^{j\phi_1}\ \&\ Z_2 = X_2 + Y_2 = |Z_2|e^{j\phi_2} \rightarrow Z_3 = Z_1 \pm Z_2 = \underbrace{(X_1 \pm X_2)}_{\mathrm{Re}\{Z_3\}=X_3} + j\underbrace{(Y_1 \pm Y_2)}_{\mathrm{Im}\{Z_3\}=Y_3}$$

$$Z_3 = Z_1 Z_2 = \underbrace{(X_1X_2 - Y_1Y_2)}_{\mathrm{Re}\{Z_3\}=X_3} + j\underbrace{(X_1Y_2 + X_2Y_1)}_{\mathrm{Im}\{Z_3\}=Y_3} = \underbrace{|Z_1||Z_2|}_{|Z_3|}e^{j\overbrace{(\phi_1+\phi_2)}^{\phi_3}}$$

$$Z_3 = \frac{Z_1}{Z_2} = \underbrace{\frac{(X_1X_2 + Y_1Y_2)}{X_2^2 + Y_2^2}}_{\mathrm{Re}\{Z_3\}=X_3} + j\underbrace{\frac{(X_2Y_1 - X_1Y_2)}{X_2^2 + Y_2^2}}_{\mathrm{Im}\{Z_3\}=Y_3} = \underbrace{(|Z_1|/|Z_2|)}_{|Z_3|}e^{j\overbrace{(\phi_1-\phi_2)}^{\phi_3}}$$

Different versions of Euler's Formula

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta \quad \text{or} \quad Me^{\pm j\theta} = M\cos\theta \pm jM\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \& \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \frac{j}{2}\left(e^{-j\theta} - e^{j\theta}\right)$$

Some Handy Complex-Number Identities

$$j = \sqrt{-1},\ j^2 = -1,\ j^3 = -j,\ j^4 = 1,\ \frac{1}{j} = -j,\ \frac{A}{B \pm jC} = \frac{A(B \mp jC)}{B^2 + C^2},\ \frac{A}{B \pm jC} \neq \frac{A}{B} \mp j\frac{A}{C}$$

$$e^{\pm j\pi/4} = (1 \pm j)/\sqrt{2},\ e^{\pm j\pi/2} = \pm j,\ e^{\pm j\pi} = -1,\ e^{\pm j3\pi/2} = \mp j,\ e^{\pm j2\pi} = 1,\ e^{\pm j3\pi} = -1$$

Roots of a Quadratic Equation

$$\left.\begin{array}{c} as^2 + bs + c = 0 \\ \text{or} \\ s^2 + \frac{b}{a}s + \frac{c}{a} = 0 \end{array}\right\} \rightarrow \text{Roots } s_1, s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a} \quad \text{where } \Delta = b^2 - 4ac$$

$$\text{Note that } s_1, s_2 = \begin{cases} \text{real \& different}, & \Delta > 0 \\ \text{real \& equal}, & \Delta = 0 \\ \text{complex conjugate}, & \Delta < 0 \end{cases} \qquad \text{Note also that } s_1 + s_2 = -\frac{b}{a} \text{ and } s_1 s_2 = \frac{c}{a}$$

Derivatives & Integrals of Basic Functions

$$a, b, \text{and } c \text{ are real constants} \rightarrow \frac{d}{dt}\left(at^{\pm b}\right) = a(\pm b)\, t^{\pm b - 1}, \quad \frac{d}{dt}\left(ae^{\pm bt}\right) = a(\pm b)e^{\pm bt}$$

$$\frac{d}{dt}\left(ae^{\pm bt \pm c}\right) = a(\pm b)e^{\pm bt \pm c}, \quad \frac{d}{dt}\left(ae^{\pm jbt \pm jc}\right) = a\left(\underbrace{\pm j}_{e^{\pm jp/2}}\, b\right)e^{\pm jbt \pm jc} = abe^{j(\pm bt \pm c \pm p/2)}$$

$$\frac{d}{dt}\left(ae^{(\pm b \pm jc)t}\right) = a\underbrace{(\pm b \pm jc)}_{|Z|e^{jf}}e^{(\pm b \pm jc)t} = a\underbrace{\sqrt{b^2 + c^2}}_{|Z|}\, e^{\pm bt + j\left(\pm ct + \underbrace{\tan^{-1}(\pm c/\pm b)}_{f}\right)}$$

$$\frac{d}{dt}\left(a\sin(bt + c)\right) = ab\cos(bt + c), \quad \frac{d}{dt}\left(a\cos(bt + c)\right) = -ab\sin(bt + c)$$

$$\int at^{\pm b}dt = \frac{a}{(\pm b + 1)}\, t^{\pm b + 1} + k, \quad \int ae^{\pm bt}dt = \left(\frac{a}{\pm b}\right)e^{bt} + k, \quad \int ae^{\pm bt \pm c}dt = \left(\frac{a}{\pm b}\right)e^{\pm bt \pm c} + k$$

$$\int ae^{\pm jbt \pm jc}dt = \left(\frac{a}{\pm jb}\right)e^{\pm jbt \pm jc} + k = \left(\frac{a}{b}\right)e^{j(\pm bt \pm c \mp p/2)} + k$$

$$\int ae^{(\pm b \pm jc)t}dt = \frac{a}{\underbrace{(\pm b \pm jc)}_{|Z|e^{jf}}}e^{(\pm b \pm jc)t} + k = \frac{a}{\underbrace{\sqrt{b^2 + c^2}}_{|Z|}}\, e^{\pm bt + j\left(\pm ct - \underbrace{\tan^{-1}(\pm c/\pm b)}_{f}\right)} + k$$

$$\int a\sin(bt + c)dt = -\frac{a}{b}\cos(bt + c) + k, \quad \int a\cos(bt + c)dt = \frac{a}{b}\sin(bt + c) + k$$

$$\frac{d}{dt}\left(a\tan(bt + c)\right) = \frac{ab}{\cos^2(bt + c)}, \quad \int a\tan(bt + c)dt = -\frac{a}{b}\ln\left(\cos(bt + c)\right) + k$$

$$\int te^{at}dt = \frac{e^{at}}{a^2}(at-1)+k, \quad \int t^2 e^{at}dt = \frac{e^{at}}{a^3}\left(a^2t^2 - 2at + 2\right)+k$$

$$\int t\sin(at)dt = \frac{1}{a^2}(\sin(at) - at\cos(at)) + k, \quad \int t\cos(at)dt = \frac{1}{a^2}(\cos(at) + at\sin(at)) + k$$

$$\int t^2 \sin(at)dt = \frac{1}{a^3}\left(2at\sin(at) + 2\cos(at) - a^2t^2\cos(at)\right) + k$$

$$\int t^2 \cos(at)dt = \frac{1}{a^3}\left(2at\cos(at) - 2\sin(at) + a^2t^2\sin(at)\right) + k$$

$$\int e^{at}\sin(bt)dt = \frac{e^{at}}{a^2+b^2}(a\sin(bt) - b\cos(bt)) + k$$

$$\int e^{at}\cos(bt)dt = \frac{e^{at}}{a^2+b^2}(a\cos(bt) + b\sin(bt)) + k$$

$$\int \frac{dt}{a^2+t^2} = \frac{1}{a}\tan^{-1}\left(\frac{t}{a}\right) + k$$