Granular Fabric and Stress at the Critical State

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Mechanical Engineering Congress and Exposition
Seattle, Washington

SETTING THE STANDARD

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Outline

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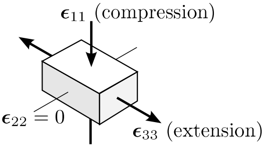
Introduction — Granular fabric

Fabric anisotropy after loading:



Plane strain biaxial loading

Introduction — Granular fabric

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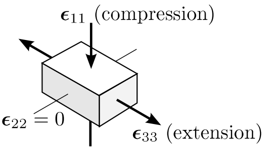

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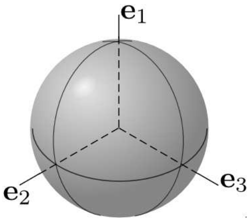

Unit sphere, Ω

Introduction — Granular fabric

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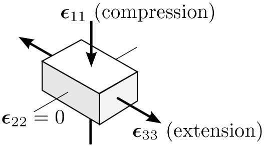

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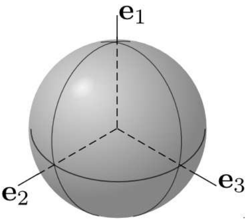

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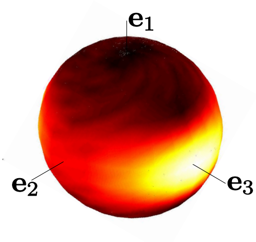

Contact density

Introduction — Granular fabric

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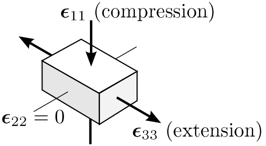

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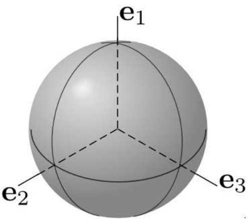

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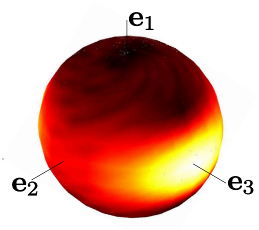


Contact density

Fabric anisotropy $\begin{cases} \text{contact density} \\ \text{contact force density} \end{cases}$

Introduction — Current study

Questions:

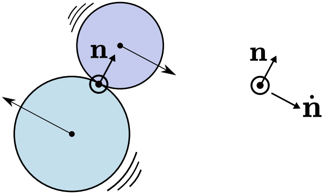
- How/why does granular fabric evolve?
- Relationship between the changing fabric and the bulk stress?

Outline

1. Introduction & Scope
2. Principles
   - Contact movements
   - Fabric evolution
3. Model & Calibration
4. Applications

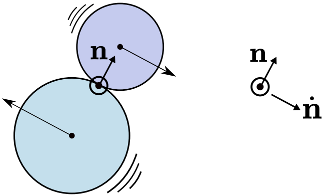
Principles — Contact movement

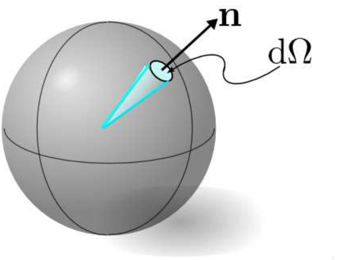
Contact movement — Movement of a unit normal vector:



Principles — Contact movement

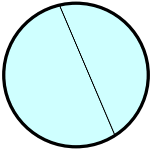
Contact movement — Movement of a unit normal vector:





Principles — Contact movement

Contact movements through $d\Omega$:



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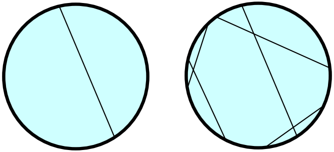

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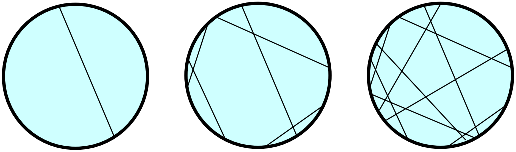

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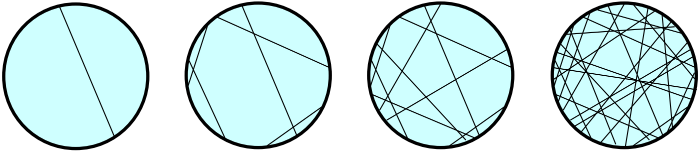
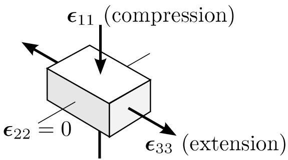


Principles — Contact movement

A "prevailing migration" of contacts ?



Principles — Contact movement

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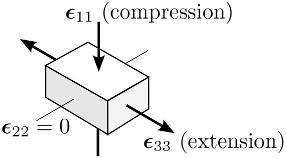
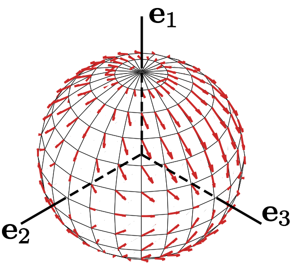




Prevailing migration rate $\dot{\mathbf{n}}(\mathbf{n})$

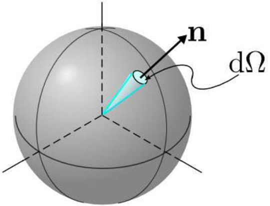
Principles — Contact movements

Prevailing contact movement — DEM results:

$$\dot{\mathbf{n}}(\mathbf{n}) \approx \underbrace{\alpha}_{\alpha = 1.5} \underbrace{(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})}_{\text{Projection tensor}} \cdot \underbrace{\overset{\text{Bulk deformation}}{\mathbf{D}} \cdot \mathbf{n}}_{\text{Movement vector}}$$

Principles — Fabric evolution

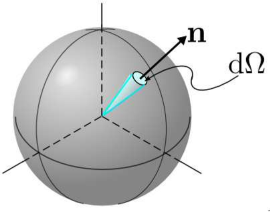
Fabric densities:



Unit sphere

Principles — Fabric evolution

Fabric densities:

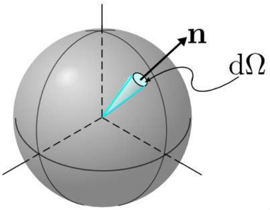


Unit sphere

$\widehat{g}(\mathbf{n})$	contact density
$\widehat{f^n}(\mathbf{n})$	normal force density
$\widehat{f^t}(\mathbf{n})$	tangential force density

Principles — Fabric evolution

Fabric densities:



Unit sphere

$\widehat{g}(\mathbf{n})$	contact density
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Example: $\widehat{g}(\mathbf{n}) =$ number of contacts per *particle* per *area* of unit sphere

Principles — Fabric evolution

Fabric evolution:

$$\left.\frac{\partial \widehat{g}(\mathbf{n})}{\partial t}\right|_{\mathbf{n}} = \color{red}{?}$$

Principles — Fabric evolution

Fabric evolution:

$$\left.\frac{\partial \widehat{g}(\mathbf{n})}{\partial t}\right|_{\mathbf{n}} = \left(\frac{\partial \widehat{g}(\mathbf{n})}{\partial t}\right)_{\text{matl}} - \boldsymbol{\nabla} \cdot \left(\dot{\mathbf{n}}\, \widehat{g}(\mathbf{n})\right) + \left(\frac{\partial \widehat{g}(\mathbf{n})}{\partial t}\right)_{\text{diffusion}}$$

$\boxed{1}$ Matl. rate

$\boxed{2}$ Divergence / convection

$\boxed{3}$ Diffusion

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$\boxed{1}$	$\boxed{2}$	$\boxed{3}$
Matl. rate	Divergence / convection	Diffusion

Ma and Zhang (2006)

Principles — Fabric evolution

Material (source) density rate:

$$\left(\frac{\partial \widehat{g}(\mathbf{n})}{\partial t}\right)_{\mathrm{matl}} \longrightarrow$$

$\boxed{1}$

Principles — Fabric evolution

Divergence & convection rates:

$$\left.\frac{\partial\widehat{g}(\mathbf{n})}{\partial t}\right|_{\mathbf{n}} = \ \cdots$$

$$\left(\frac{\partial\widehat{g}(\mathbf{n})}{\partial t}\right)_{\text{matl}} - (\nabla\cdot\dot{\mathbf{n}})\,\widehat{g}(\mathbf{n}) \; - \dot{\mathbf{n}}\cdot\left(\nabla\widehat{g}(\mathbf{n})\right) \; + \left(\frac{\partial\widehat{g}(\mathbf{n})}{\partial t}\right)_{\text{diffusion}}$$

$\boxed{1}$	$\boxed{2a}$	$\boxed{2b}$	$\boxed{3}$
Matl. rate	Divergence	Convection	Diffusion

Principles — Fabric evolution

Contact divergence rate:

$$(\boldsymbol{\nabla} \cdot \dot{\mathbf{n}})\,\widehat{g}(\mathbf{n}) \qquad \longrightarrow$$

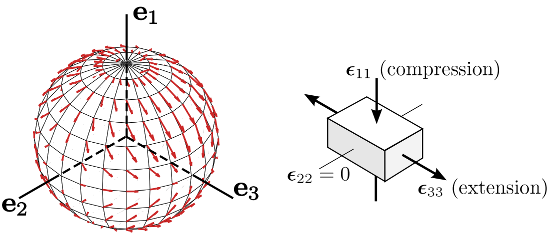
$\boxed{\text{2a}}$

Principles — Fabric evolution

Contact divergence rate:

$$(\nabla \cdot \dot{\mathbf{n}})\,\widehat{g}(\mathbf{n}) \longrightarrow$$

$\boxed{2a}$



Principles — Fabric evolution

Contact convection rate:

$$\dot{\mathbf{n}} \cdot \left(\nabla \widehat{g}(\mathbf{n})\right) \quad \longrightarrow$$

$\boxed{\text{2b}}$

Principles — Fabric evolution

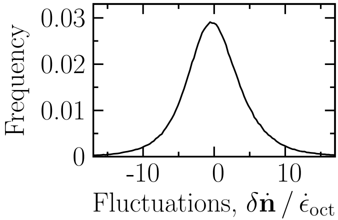
Contact convection rate:

$$\dot{\mathbf{n}} \cdot \left(\nabla \widehat{g}(\mathbf{n})\right) \longrightarrow$$

$\boxed{\text{2b}}$

Principles — Fabric evolution

Contact diffusion due to velocity fluctuations — DEM results:



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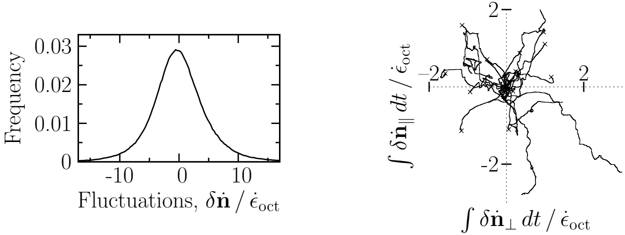

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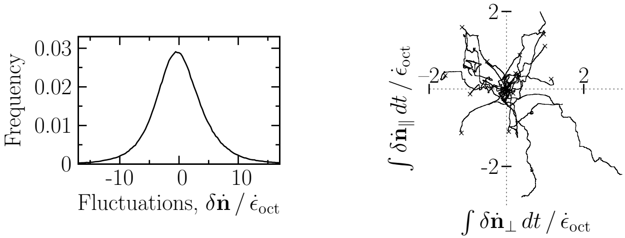

$$\left(\frac{\partial \widehat{g}(\mathbf{n})}{\partial t}\right)_{\text{diffusion}} = D_g\, \nabla^2 \widehat{g}(\mathbf{n})\, \dot{\epsilon}\ , \qquad D_g = 0.03$$

$\boxed{3}$

Outline

Calibration — force rates

Evolution of normal force density:

$$\left.\frac{\partial \widehat{f^{\mathrm{n}}}(\mathbf{n})}{\partial t}\right|_{\mathbf{n}} = \left(\frac{\partial \widehat{f^{\mathrm{n}}}(\mathbf{n})}{\partial t}\right)_{\text{matl}} - \boldsymbol{\nabla}\cdot\left(\dot{\mathbf{n}}\,\widehat{f^{\mathrm{n}}}(\mathbf{n})\right) + \left(\frac{\partial \widehat{f^{\mathrm{n}}}(\mathbf{n})}{\partial t}\right)_{\text{diffusion}}$$

$\boxed{1}$	$\boxed{2}$	$\boxed{3}$
Matl. rate	Divergence / convection	Diffusion

Calibration — force rates

Normal force material rate — DEM results:

$$\left(\frac{\partial \widehat{f^{\mathrm{n}}}}{\partial t}\right)_{\mathrm{matl}} \approx \beta \quad k^{\mathrm{n}} \quad \underbrace{\mathbf{n}\,(\overset{\text{Bulk deformation}}{\mathbf{D}} \cdot \mathbf{n})}_{\text{Mean-field rate}}$$

$$k^{\mathrm{n}} = \underbrace{\left[\frac{3G^2\overline{\ell}^4}{(1-\nu)^2}\,\widehat{f^{\mathrm{n}}}(\mathbf{n})\,\left(\widehat{g}(\mathbf{n})\right)^2\right]^{1/3}}_{\text{Hertz stiffness}}$$

$$\beta \approx 0.0024$$

Outline

Possible applications:

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1) Effect of the intermediate principal stress
2) Non-coaxial stress & strain increments
3) Effect of spatial gradients of strain
4) Softening / hardening rates

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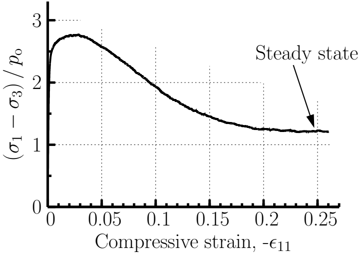
Application — Intermediate principal stress

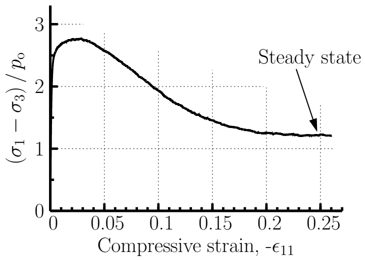
Effect of the intermediate principal stress — DEM results:



Plane strain biaxial compression

Application — Intermediate principal stress

Effect of the intermediate principal stress — DEM results:



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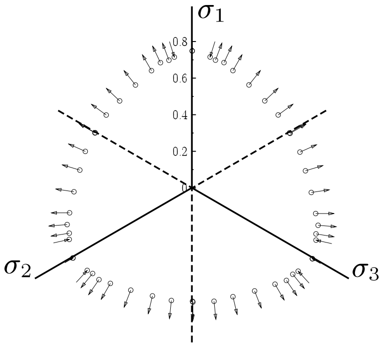


Application — Intermediate principal stress

At the steady state $\longrightarrow$ stationary densities:

$$\left.\frac{\partial \widehat{g}(\mathbf{n})}{\partial t}\right|_{\mathbf{n}} = 0$$

$$\left.\frac{\partial \widehat{f^{\mathrm{n}}}(\mathbf{n})}{\partial t}\right|_{\mathbf{n}} = 0$$

$$\left.\frac{\partial \widehat{f^{\mathrm{t}}}(\mathbf{n})}{\partial t}\right|_{\mathbf{n}} = 0$$

Application — Intermediate principal stress

$$
\begin{aligned}
\left.\frac{\partial \widehat{f^{\mathrm{n}}}(\mathbf{n})}{\partial t}\right|_{\mathbf{n}} = \; & 0.0024 \left[\frac{3G^2\overline{\ell}^4}{(1-\nu)^2} \widehat{f^{\mathrm{n}}}(\mathbf{n}) \left(\widehat{g}(\mathbf{n})\right)^2 \right]^{1/3} \mathbf{n}\cdot(\mathbf{D}\cdot\mathbf{n}) \\
& + 0.68\, \widehat{f^{\mathrm{n}}}(\mathbf{n}) \frac{\left|1.5\ (\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\cdot(\mathbf{D}\cdot\mathbf{n})\right|^2}{\dot{\epsilon}} \\
& + (3\times 1.5)\,\mathbf{n}\cdot(\mathbf{D}\cdot\mathbf{n})\ \widehat{f^{\mathrm{n}}}(\mathbf{n}) \\
& - 1.5\ (\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\cdot(\mathbf{D}\cdot\mathbf{n})\cdot\left(\boldsymbol{\nabla}\widehat{f^{\mathrm{n}}}(\mathbf{n})\right) \\
& + 0.03\left(\nabla^2\widehat{f^{\mathrm{n}}}(\mathbf{n})\right)\ \dot{\epsilon} \\
= \; & 0
\end{aligned}
$$

Application — Intermediate principal stress

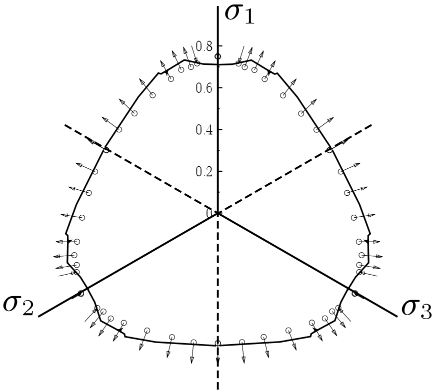
At the steady state $\longrightarrow$ stationary densities:

$$\left.\frac{\partial \widehat{g}(\mathbf{n})}{\partial t}\right|_{\mathbf{n}} = 0\,, \qquad \left.\frac{\partial \widehat{f^{\mathrm{n}}}(\mathbf{n})}{\partial t}\right|_{\mathbf{n}} = 0\,, \qquad \left.\frac{\partial \widehat{f^{\mathrm{t}}}(\mathbf{n})}{\partial t}\right|_{\mathbf{n}} = 0$$

- Coupled non-linear PDEs on the 2D unit sphere
- Solution depends on the “data” D: the bulk deformation
- Solve for the density distributions $\widehat{g}(\mathbf{n})$, $\widehat{f^{\mathrm{n}}}(\mathbf{n})$, and $\widehat{f^{\mathrm{t}}}(\mathbf{n})$
- Use $\widehat{f^{\mathrm{n}}}(\mathbf{n})$ and $\widehat{f^{\mathrm{t}}}(\mathbf{n})$ to find the corresponding stress $\boldsymbol{\sigma}$

Application — Intermediate principal stress

Solution of PDEs:



Conclusions

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- A model for fabric evolution
- Based on transport of contact and force densities
- Calibrated with DEM
- Reasonable prediction of effect of intermediate principal stress

Questions?