Micro-Mechanics of Granular Flow at Large Strains

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Engineering Mechanics Institute
Inaugural International Conference
Minneapolis, Minnesota

em08
The INAUGURAL INTERNATIONAL CONFERENCE
of the ENGINEERING MECHANICS INSTITUTE

May 18–22, 2008

Introduction

Granular “genre”:

Dense packings
Slow loading

Granular phenomena at large strains:

- Reduced stiffness
- Softening
- Localized deformation
- Non-coaxiality, $d\boldsymbol{\sigma} \leftrightarrow d\boldsymbol{\epsilon}$
- Gradient effects, $d\boldsymbol{\sigma} \leftrightarrow \nabla_{x}\boldsymbol{\epsilon}$

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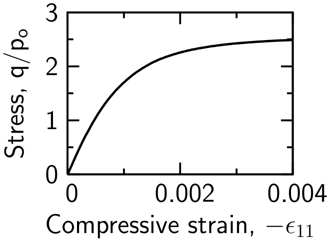

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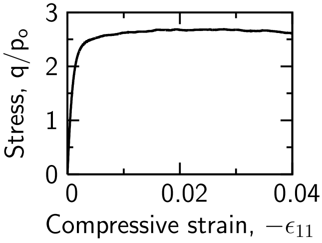

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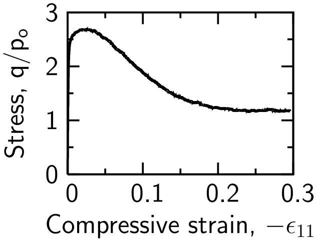


Three views of large-strain micro-mechanics

Three approaches to micro-mechanics:

1. Discrete contact approach
2. Contact distribution approach
3. Discrete stiffness approach

Focus on

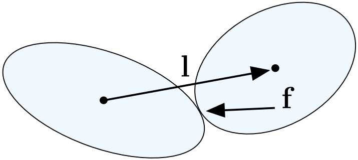
- Incremental behavior
- "Mechanical" vs. "Geometric" effects
- Large vs. small strains

Outline

1. Micro-view 1: Discrete contact approach
   - Average stress
   - Example
2. Micro-view 2: Contact distribution approach
3. Micro-view 3: Discrete stiffness approach

Discrete contact approach

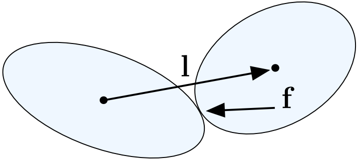
Calculation of average stress



$$\boldsymbol{\sigma} = \frac{1}{V}\sum \mathbf{l} \otimes \mathbf{f}$$

Discrete contact approach

Calculation of stress increment

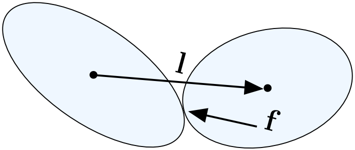


$$\boldsymbol{\sigma} = \frac{1}{V}\sum \mathbf{l} \otimes \mathbf{f}$$

$$d\boldsymbol{\sigma} = -\frac{dV}{V}\boldsymbol{\sigma} + \underbrace{\frac{1}{V}\sum \mathbf{l} \otimes d\mathbf{f}}_{\text{Mechanical}} + \underbrace{\frac{1}{V}\sum d\mathbf{l} \otimes \mathbf{f}}_{\text{Geometric}}$$

Discrete contact approach

Calculation of stress increment



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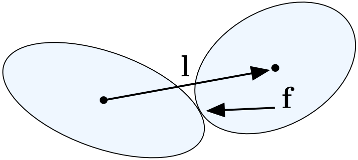
Discrete contact approach

Calculation of stress increment



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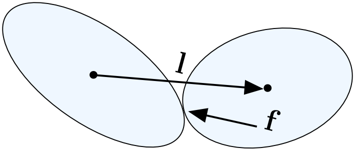
Discrete contact approach

Calculation of stress increment



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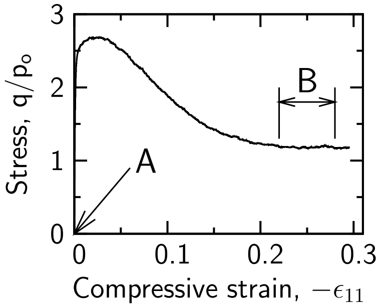
Discrete contact approach — Example

Stress rates during loading, $d\sigma \,/\, d\epsilon$

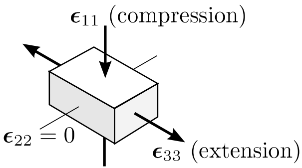


	A	B
Mechanical	$+2,253$	$+4.1$
Geometric	-1.9	-4.1
$\sum =$	$+2,251$	0

Outline

Contact distribution approach

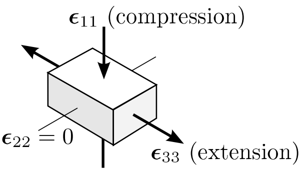
Contact force anisotropy during loading:



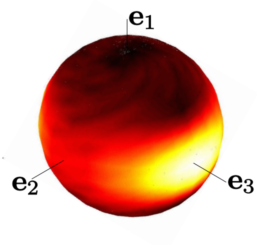
Plane strain biaxial loading

Contact distribution approach

Contact force anisotropy during loading:



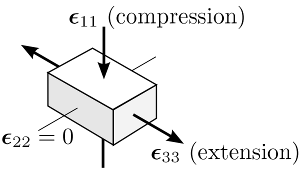
Plane strain biaxial loading



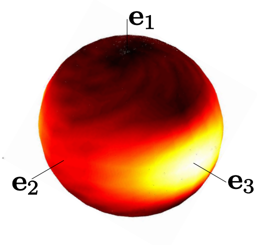
Force density

Contact distribution approach

Contact force anisotropy during loading:



Plane strain biaxial loading



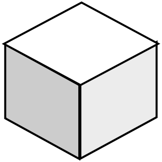
Force density

Density $\widehat{\mathbf{f}}(\mathbf{n})$

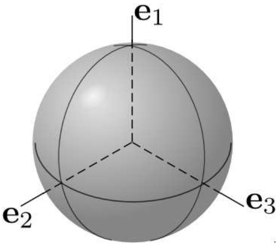
Density rate?

Contact distribution approach

Migration of contacts?



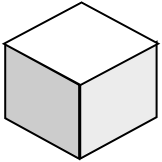
Assembly box



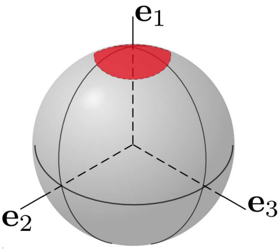
Unit sphere, Ω

Contact distribution approach

Migration of contacts?



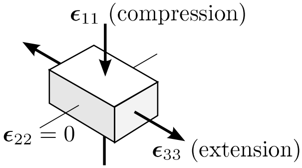
Assembly box



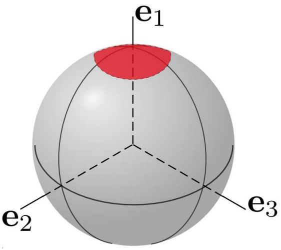
Contact set

Contact distribution approach

Migration of contacts?



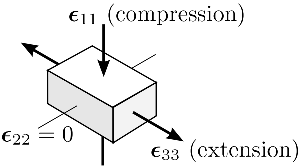
Plane strain biaxial loading



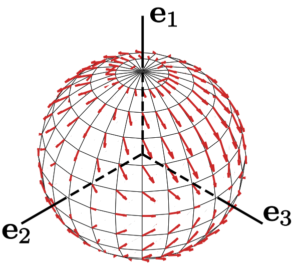
Contact set

Contact distribution approach

Migration of contacts?



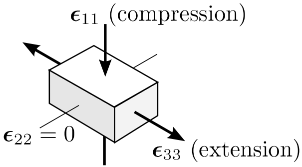
Plane strain biaxial loading



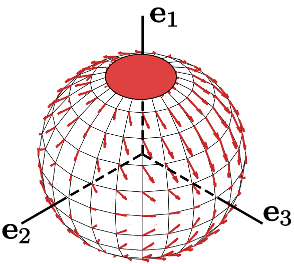
Prevailing migration $\dot{\mathbf{n}}(\mathbf{n})$

Contact distribution approach

Migration of contacts?

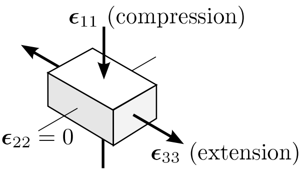


Plane strain biaxial loading



Contact distribution approach

Migration of contacts?



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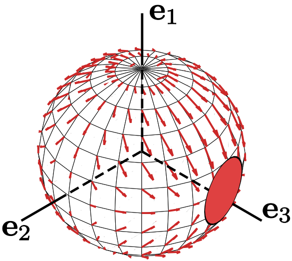
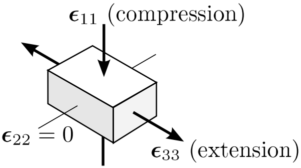


Contact distribution approach

Migration of contacts?



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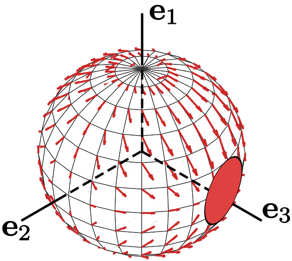


Stress & force evolution

Force distribution rate:

$$\left.\frac{\partial \widehat{\mathbf{f}}(\mathbf{n})}{\partial t}\right|_{\mathbf{n}} = \left(\frac{\partial \widehat{\mathbf{f}}(\mathbf{n})}{\partial t}\right)_{\text{matl}} - \nabla \cdot \left(\dot{\mathbf{n}}\,\widehat{\mathbf{f}}(\mathbf{n})\right) + \left(\frac{\partial \widehat{\mathbf{f}}(\mathbf{n})}{\partial t}\right)_{\text{diffusion}}$$

1 Matl. rate

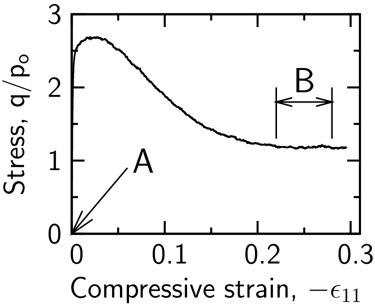
2 Divergence / convection

3 Diffusion

Ma and Zhang (2006)

Contact distribution approach — Example

Stress rates during loading, $d\boldsymbol{\sigma} \,/\, d\epsilon$



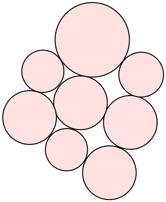
	A	B
Mechanical	$+3,510$	$+4.0$
Geometric	-2	-4.0
$\sum =$	$+3,508$	0

Outline

1. Micro-view 1: Discrete contact approach
2. Micro-view 2: Contact distribution approach
3. Micro-view 3: Discrete stiffness approach
   - Incremental stiffness
   - Examples
   - Pathologies I
   - Pathologies II

Discrete stiffness approach

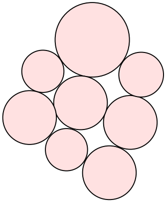
Incremental stiffness of a particle assembly:



Particle movements		External forces
$[\, d\mathbf{u} \,]$	$\Rightarrow$	$[\, d\mathbf{f} \,]$

Discrete stiffness approach

Incremental stiffness of a particle assembly:



$$[\,\mathbf{K}\,][\,d\mathbf{u}\,] = [\,d\mathbf{f}\,]$$

Discrete stiffness approach

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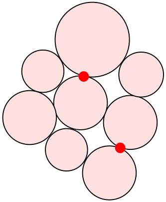

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Discrete stiffness approach

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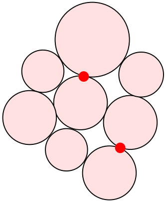
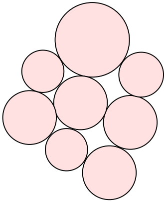


$$[\,\mathbf{K}^{\bullet,\bullet}\,][\,d\mathbf{u}\,] = [\,d\mathbf{f}\,]$$
$$[\,\mathbf{K}^{\circ,\bullet}\,][\,d\mathbf{u}\,] = [\,d\mathbf{f}\,]$$
$$[\,\mathbf{K}^{\bullet,\circ}\,][\,d\mathbf{u}\,] = [\,d\mathbf{f}\,]$$
$$[\,\mathbf{K}^{\circ,\circ}\,][\,d\mathbf{u}\,] = [\,d\mathbf{f}\,]$$

Discrete stiffness approach

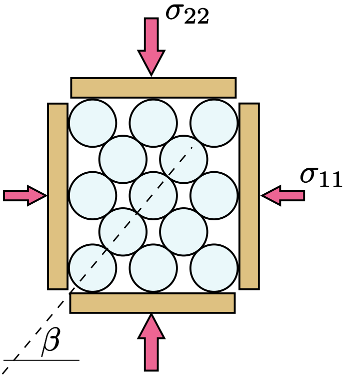
Incremental stiffness of a particle assembly:

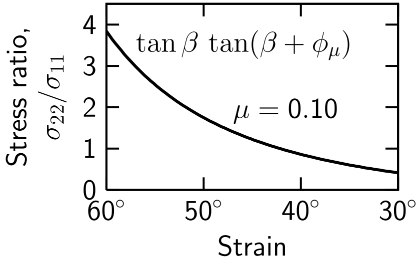


$$\left([\,\mathbf{K}^{\text{Mech.}}] + [\,\mathbf{K}^{\text{Geom.}}] \right) [\,d\mathbf{u}\,] \; = \; [\,d\mathbf{f}\,]$$

Discrete stiffness approach — Example I

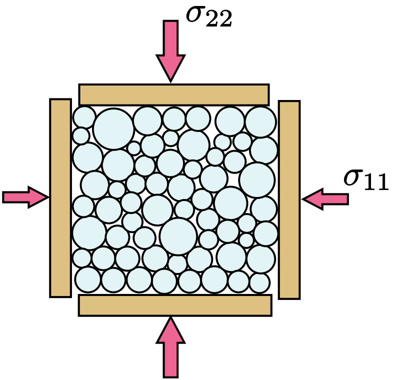
Example I: Regular array

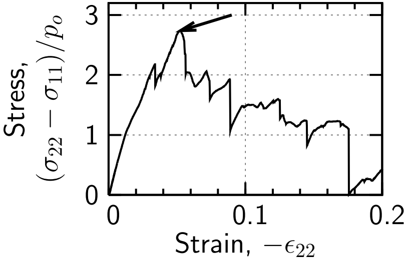




Discrete stiffness approach — Example II

Example II: Assembly of 64 disks





Discrete stiffness approach — Example II

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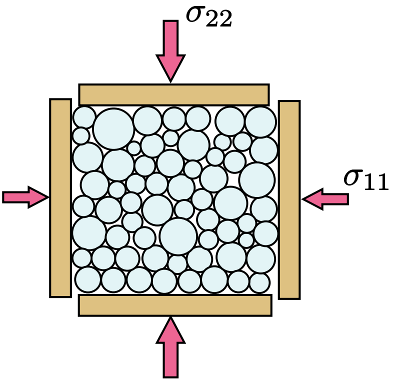
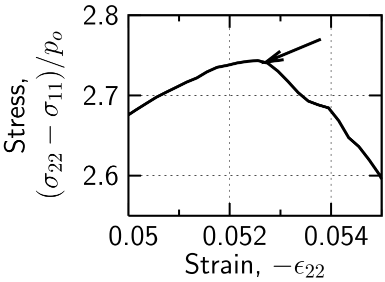
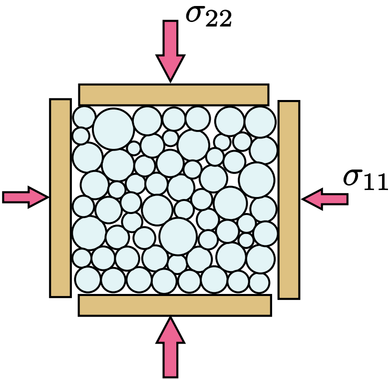
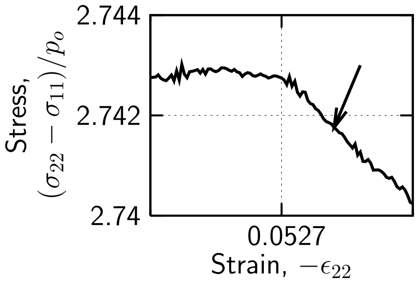




Discrete stiffness approach — Example II

Example II: Assembly of 64 disks





Discrete stiffness approach

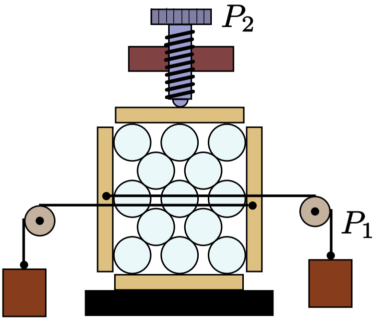
$$[\,\mathbf{K}\,][\,d\mathbf{u}\,] = [\,d\mathbf{f}\,]$$

Stiffness “pathologies”:

1. Softening
2. External instability
3. Internal instability
4. Non-uniqueness / bifurcation

Discrete stiffness approach — Example I

Pathology 1: Softening



Softening

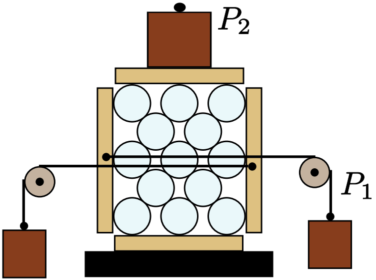
$$\delta^2 W = dP_1 d\ell_1 + dP_2 d\ell_2 < 0$$

Externally stable

Softening is entirely geometric!

Discrete stiffness approach — Example I

Pathology 2: External instability



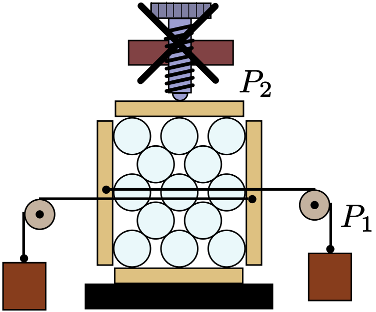
Externally unstable

$$\delta^2 W = dP_1 d\ell_1 + dP_2 d\ell_2 < 0$$

Collapse

Discrete stiffness approach — Example I

Pathology 3: Internal instability



Externally stable

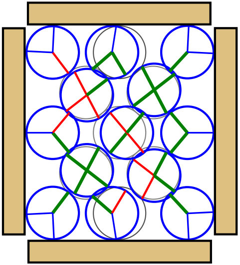
Internally unstable

$$\delta^2 W = [\, d\mathbf{u} \,]^T \, [\, \mathbf{K} \,] \, [\, d\mathbf{u} \,] < 0$$

Discrete stiffness approach — Example I

Pathology 3: Internal instability

Example instability mode



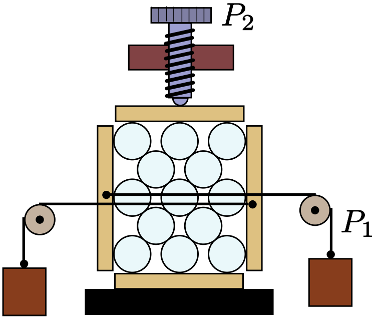
Externally stable

Internally unstable

$$\delta^2 W = [\, d\mathbf{u} \,]^T \, [\, \mathbf{K} \,] \, [\, d\mathbf{u} \,] < 0$$

Discrete stiffness approach — Example I

Pathology 4: Non-uniqueness / bifurcation



Softening

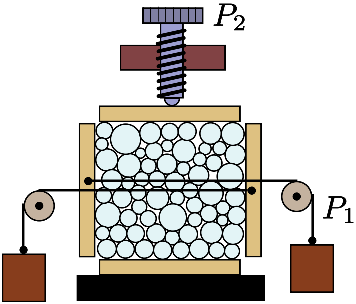
$$\delta^2 W = dP_1 d\ell_1 + dP_2 d\ell_2 < 0$$

Externally stable

Solution is unique !

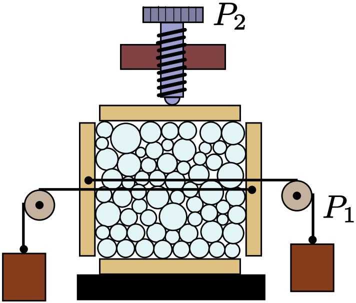
Discrete stiffness approach — Example II

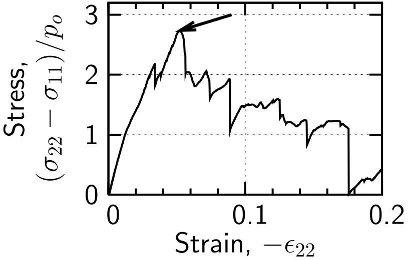
Example II: Assembly of 64 disks



Discrete stiffness approach — Example II

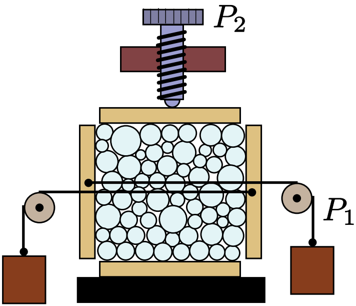
Example II: Assembly of 64 disks





Discrete stiffness approach — Example II

Example II: Assembly of 64 disks



(1) Softening

$$\delta^2 W = dP_1 d\ell_1 + dP_2 d\ell_2 \; < \; 0$$

Softening is geometric

(2) Externally stable

(3) Internally unstable

At least six unstable modes

$$\delta^2 W = [\, d\mathbf{u} \,]^T \, [\, \mathbf{K} \,] \, [\, d\mathbf{u} \,] < 0$$

(4) Unique solution

Conclusions

Questions?