Scaling in Granular Materials

Matthew R. Kuhn

University of Portland

Powders & Grains 2005

Outline

Introduction

Limitations of the talk!

- Quasi-static (time-invariant) behavior of dense packings
- Durable particles
- Experimental & analytical
- Discrete micro-mechanics
- Emphasis on behavior at large strains

Five Examples of Granular Behavior

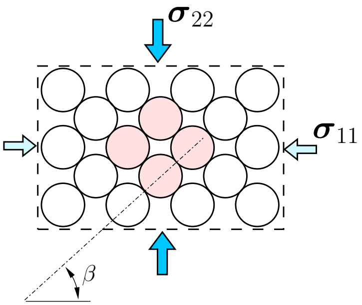
Dominant behaviors at large strains:

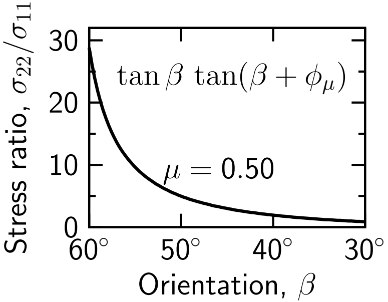
- Softening (2 examples)
- Instability (2 examples)
- Localization (1 example)

Outline

Example #1: Granular Softening

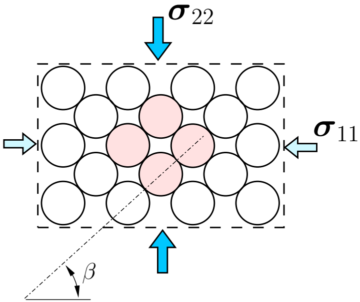
Softening of a regular 2D array

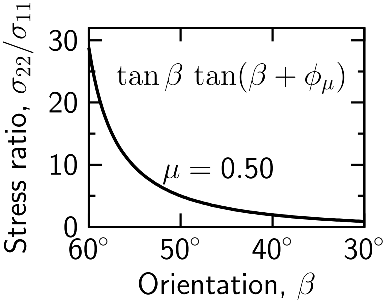




Example #1: Granular Softening

Softening of a regular 2D array





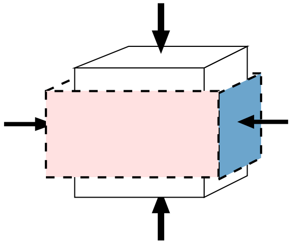
$$\int_{\partial V} (d\mathbf{u} \cdot d\mathbf{f}) < 0 \quad \text{or} \quad [d\mathbf{u}]^{\mathsf{T}} [d\mathbf{f}] < 0$$

Example #2: Granular Softening

Softening of 4096 spheres — DEM simulation

Densely packed

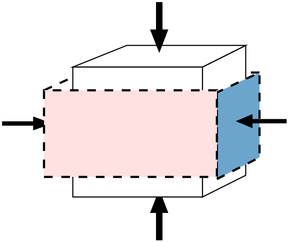
Plane-strain, biaxial compression

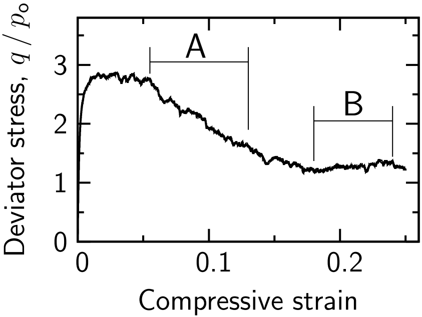


Example #2: Granular Softening

Softening of 4096 spheres — DEM simulation

Plane-strain, biaxial compression

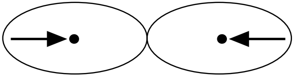




Outline

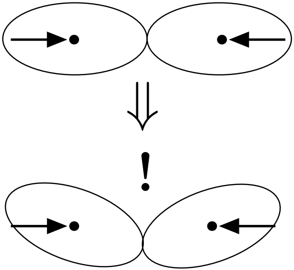
Example #3: Instability

Instability of 2-particle systems



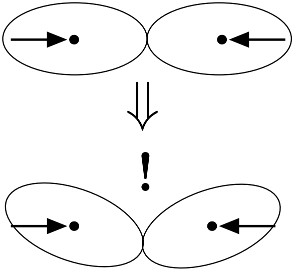
Example #3: Instability

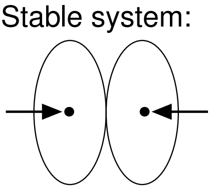
Instability of 2-particle systems



Example #3: Instability

Instability of 2-particle systems

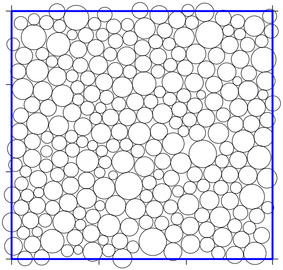




Example #4: Instability

Instability in 256 disks — DEM simulation

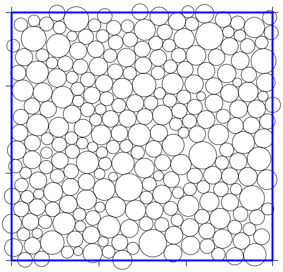
Biaxial compression

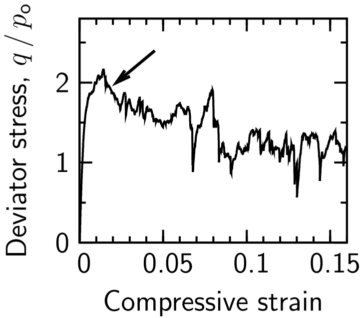


Example #4: Instability

Instability in 256 disks — DEM simulation

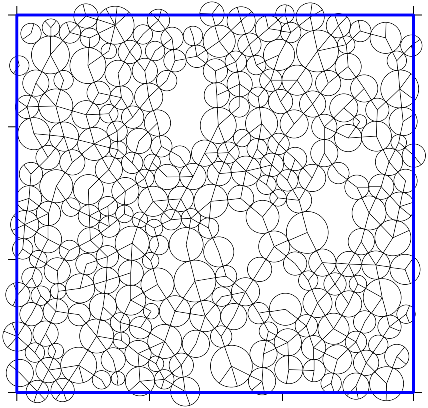
Biaxial compression





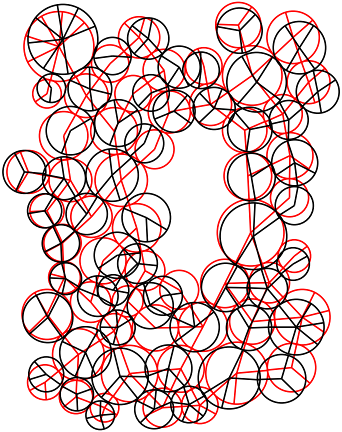
Example #4: Instability

256 disks — DEM simulation



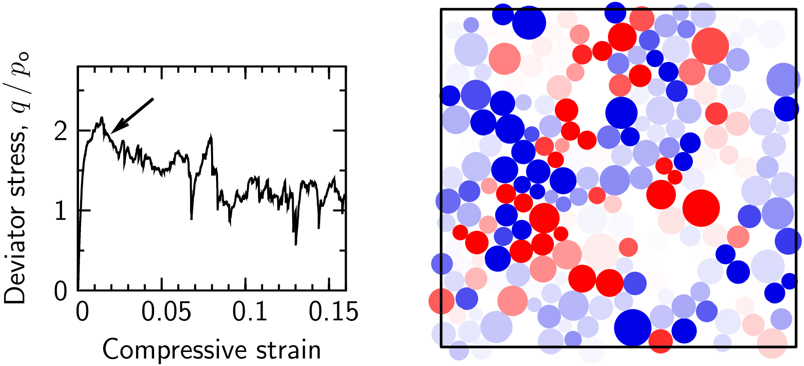
Example #4: Instability

256 disks — Detail around a granular “hole”



Example #4: Instability

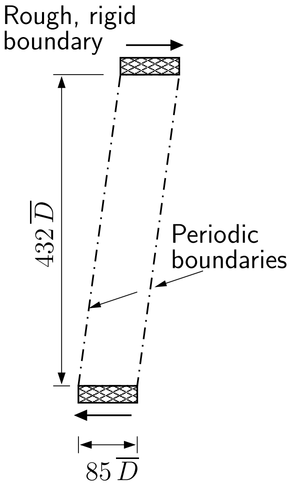
256 disks — Plot of internal instability



Outline

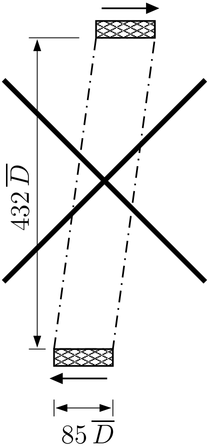
Example #5: Localization

Localization in 40,500 disks — DEM simulation



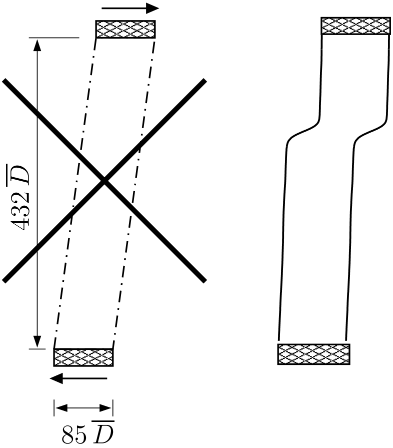
Example #5: Localization

40,500 disks — Localized shearing



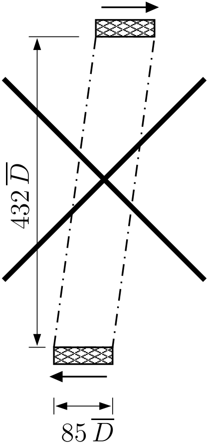
Example #5: Localization

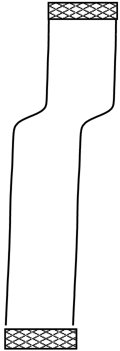
40,500 disks — Localized shearing

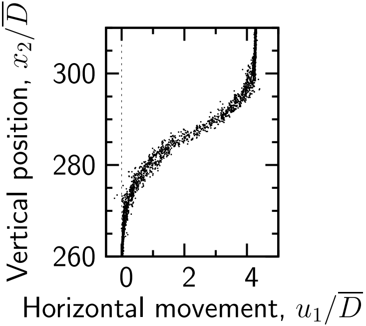


Example #5: Localization

40,500 disks — Localized shearing







Outline

Granular Softening

Origins of granular softening:

1) Mechanical

Produced by contact deformations

Depends upon particle material properties

Granular Softening

Origins of granular softening:

1) Mechanical

Produced by contact deformations

Depends upon particle material properties

2) Geometric

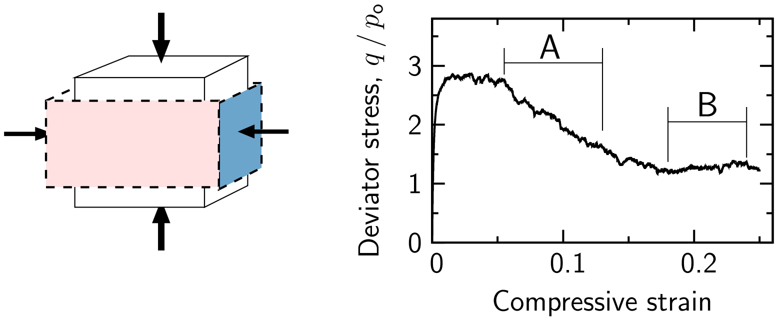
Produced by contact re-orientations

Depends upon particle shapes

Example: Granular Softening

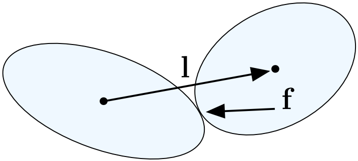
Softening of 4096 spheres — DEM results

Plane-strain, biaxial compression



Granular Softening

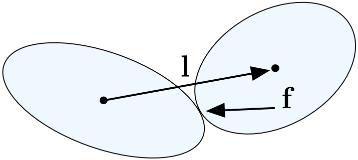
Calculation of average stress



$$\boldsymbol{\sigma} = \frac{1}{V}\sum \mathbf{l} \otimes \mathbf{f}$$

Granular Softening

Calculation of stress increment

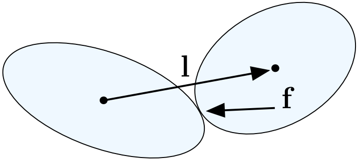


$$\boldsymbol{\sigma} = \frac{1}{V} \sum \mathbf{l} \otimes \mathbf{f}$$

$$d\boldsymbol{\sigma} = -\frac{dV}{V}\boldsymbol{\sigma} + \underbrace{\frac{1}{V} \sum \mathbf{l} \otimes d\mathbf{f}}_{\text{Mechanical}} + \underbrace{\frac{1}{V} \sum d\mathbf{l} \otimes \mathbf{f}}_{\text{Geometric}}$$

Granular Softening

Calculation of stress increment

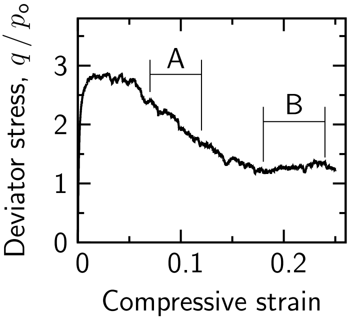


$$\boldsymbol{\sigma} = \frac{1}{V} \sum \mathbf{l} \otimes \mathbf{f}$$

$$d\boldsymbol{\sigma} = -\frac{dV}{V}\boldsymbol{\sigma} + \underbrace{\frac{1}{V} \sum \mathbf{l} \otimes d\mathbf{f}}_{\text{Mechanical}} + \underbrace{\frac{1}{V} \sum d\mathbf{l} \otimes \mathbf{f}}_{\text{Geometric}}$$

Granular Softening — Example

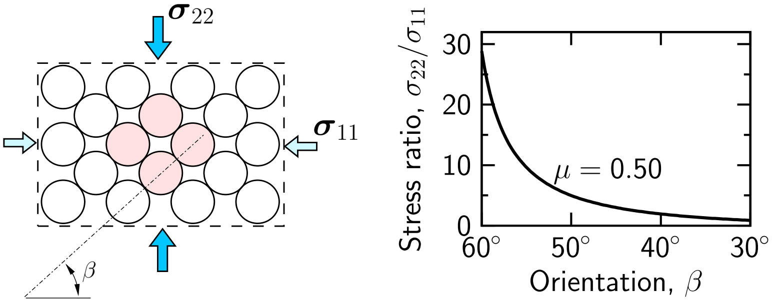
Stress rates during loading, $d\sigma / k\, d\epsilon \times 1000$



	A	B
Mechanical	−2.9	1.4
Geometric	−1.7	−1.4
$\sum =$	-4.6	0

Granular Softening — Another Example

Softening of a regular array of disks



Here, softening is entirely *geometric* !

Outline

Granular Instability

Origins of granular stiffness:

1) Mechanical

2) Geometric

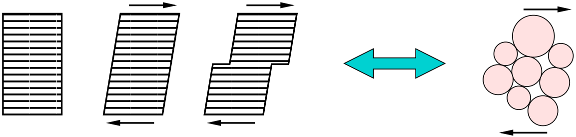
Granular Instability

Origins of granular stiffness:

1) Mechanical

2) Geometric

Example:



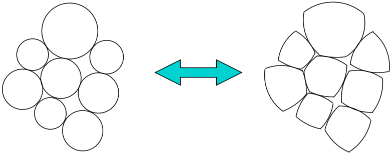
Granular Instability

Origins of granular stiffness:

1) Mechanical

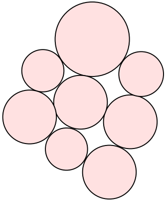
2) Geometric

Another example:



Granular Stiffness and Instability

Incremental stiffness of a particle assembly:



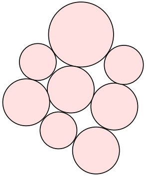
Particle movements		External forces & moments
$\begin{bmatrix} d\mathbf{u} \\ \hdashline d\theta \end{bmatrix}$	$\Rightarrow$	$\begin{bmatrix} d\mathbf{f} \\ \hdashline d\mathbf{m} \end{bmatrix}$

Contact model: soft contacts, time invariant

GEM, Y. Kishino, 1989

Granular Stiffness and Instability

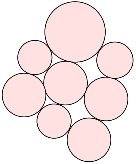
Incremental stiffness matrix:



$$\left[\quad \mathbf{K} \quad\right] \begin{bmatrix} d\mathbf{u} \\ \hdashline d\theta \end{bmatrix} = \begin{bmatrix} d\mathbf{f} \\ \hdashline d\mathbf{m} \end{bmatrix}$$

Granular Stiffness and Instability

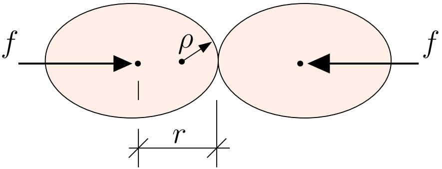
Incremental stiffness matrix:



$$\left(\left[\mathbf{K}^{\text{Mechanical}} \right] + \left[\mathbf{K}^{\text{Geometric}} \right] \right) \begin{bmatrix} d\mathbf{u} \\ \hdashline d\boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} d\mathbf{f} \\ \hdashline d\mathbf{m} \end{bmatrix}$$

Bagi 2005; Kuhn & Chang 2005

Granular Stiffness — Example



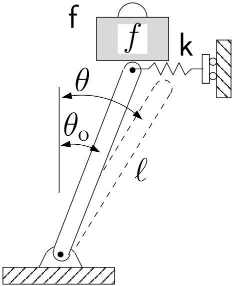
2-particle example,

6×6 stiffness matrix

$$
\underbrace{k \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & r & 0 & -1 & r \\ 0 & r & r^2 & 0 & -r & r^2 \\ \hline -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -r & 0 & 1 & -r \\ 0 & r & r^2 & 0 & -r & r^2 \end{array}\right]}_{\text{Mechanical}} + \underbrace{\frac{f}{2\rho} \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \rho - r & 0 & 1 & \rho - r \\ 0 & \rho - r & \rho^2 - r^2 & 0 & r - \rho & (\rho - r)^2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & r - \rho & 0 & -1 & r - \rho \\ 0 & \rho - r & (\rho - r)^2 & 0 & r - \rho & \rho^2 - r^2 \end{array}\right]}_{\text{Geometric}}
$$

Structural Stiffness — Analogy

Mechanical and geometric stiffnesses of a simple structure:



$$[\,\mathbf{K}\,][\,d\boldsymbol{\theta}\,] = [\,d\mathbf{f}\,]$$

$$\underbrace{\left[k\left(\sin\theta_o + \frac{\cos 2\theta}{\sin\theta}\right)}_{\text{Mechanical}} \underbrace{- \frac{f}{\ell}\cot\theta\right]}_{\text{Geometric}} \ell\, d\theta \;=\; df$$

Granular Stiffness — Scaling

Scaling of granular stiffness(es):

$$\begin{array}{llll} \text{Mechanical} & \longrightarrow \quad k & \xrightarrow{\text{Hertz}} & p^{1/3},\, E^{2/3},\, r \\ \text{Geometric} & \longrightarrow \quad f/\rho & \longrightarrow & p^{1},\, r^{2},\, 1/\rho \end{array}$$

Mechanical stiffness dominates at small strains

Both stiffnesses are important at large strains!

Granular Stiffness — Scaling

Scaling of granular stiffness(es):

$$\text{Mechanical} \longrightarrow k \xrightarrow{\text{Hertz}} p^{1/3},\ E^{2/3},\ r$$

$$\text{Geometric} \longrightarrow f/\rho \longrightarrow p^{1},\ r^{2},\ 1/\rho$$

Mechanical stiffness dominates at small strains

Both stiffnesses are important at large strains!

Granular Stiffness — Scaling

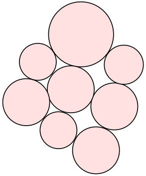
Scaling of granular stiffness(es):

$$\begin{array}{llccc} \text{Mechanical} & \longrightarrow & k & \xrightarrow{\text{Hertz}} & p^{1/3},\, E^{2/3},\, r \\ \text{Geometric} & \longrightarrow & f/\rho & \longrightarrow & p^{1},\, r^{2},\, 1/\rho \end{array}$$

Mechanical stiffness dominates at small strains

Both stiffnesses are important at large strains!

Instability and Softening — Criteria



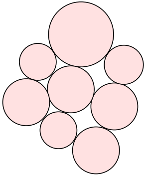
$$\left(\left[\; \mathbf{K}^{\text{Mechanical}} \; \right] + \left[\; \mathbf{K}^{\text{Geometric}} \; \right] \right) \left[\frac{d\mathbf{u}}{d\theta} \right] = \left[\frac{d\mathbf{f}}{d\mathbf{m}} \right]$$

Second-order work criteria for discrete systems:

$$\delta^2 W = \left[\frac{d\mathbf{u}}{d\theta} \right]^{\mathsf{T}} \left[\frac{d\,\mathbf{f}}{d\,\mathbf{m}} \right] < 0$$

1) Necessary for instability
2) Sufficient for softening in the direction $[d\mathbf{u}/d\theta]$

Instability and Softening — Criteria



$$\left(\left[\mathbf{K}^{\text{Mechanical}} \right] + \left[\mathbf{K}^{\text{Geometric}} \right] \right) \left[\frac{d\mathbf{u}}{d\theta} \right] = \left[\frac{d\mathbf{f}}{d\mathbf{m}} \right]$$

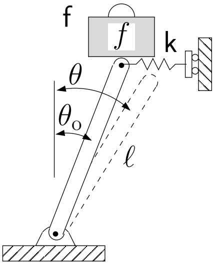
Second-order work criteria for discrete systems:

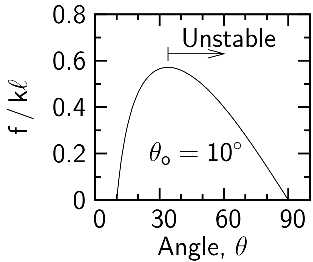
$$\delta^2 W = \left[\frac{d\mathbf{u}}{d\theta} \right]^{\mathsf{T}} \left[\frac{d\,\mathbf{f}}{d\,\mathbf{m}} \right] < 0$$

1) Necessary for instability
2) Sufficient for softening in the direction $[d\mathbf{u}/d\theta]$

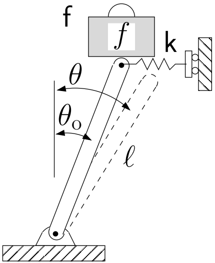
Darve et al., PG2005

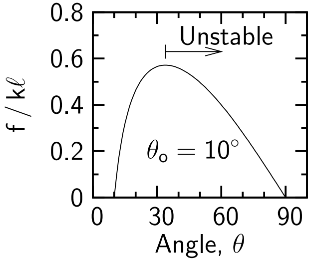
Structural Instability and Softening — Analogy





Structural Instability and Softening — Analogy

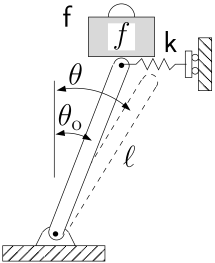


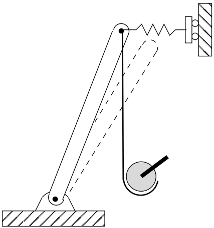


Second-order work criteria:

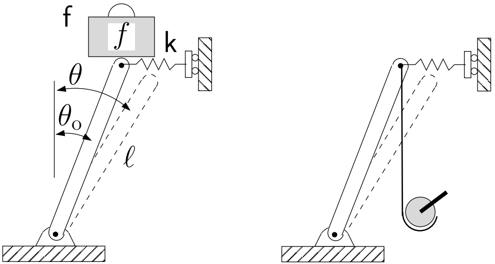
$$\delta^2 W = \ell \, d\boldsymbol{\theta} \cdot d\mathbf{f} < 0 \quad \Rightarrow$$ Unstable and Softening

Structural Instability and Softening — Analogy





Structural Instability and Softening — Analogy



Second-order work criteria:

$$\delta^2 W = \ell\, d\theta \cdot d\mathbf{f} < 0 \quad \Rightarrow$$ Stable but Softening

Granular Instability and Softening

Investigating granular instability, $\delta^2 W \overset{?}{<} 0$:

Discrete systems $\longrightarrow$ Search eigenvalues of $[\,\mathbf{K}\,]^{\text{Symmetric}}$

Difficulties:

1) Non-symmetric, $[\mathbf{K}] = [\,\mathbf{K}^{\text{Mechanical}}\,] + [\,\mathbf{K}^{\text{Geometric}}\,]$

2) Incrementally non-linear $\Rightarrow$ Multiple stiffness branches,

$$[\mathbf{K}] = \{\,[\mathbf{K}^1],\ [\mathbf{K}^2],\ \ldots\,\}$$

Must check multiple branches to investigate directional stability.

Granular Instability and Softening

Investigating granular instability, $\delta^2 W \overset{?}{<} 0$:

Discrete systems $\longrightarrow$ Search eigenvalues of $[\,\mathbf{K}\,]^{\text{Symmetric}}$

Difficulties:

1) Non-symmetric, $[\mathbf{K}] = [\,\mathbf{K}^{\text{Mechanical}}\,] + [\,\mathbf{K}^{\text{Geometric}}\,]$

2) Incrementally non-linear $\Rightarrow$ Multiple stiffness branches,

$$[\,\mathbf{K}\,] = \{\,[\,\mathbf{K}^1],\ [\,\mathbf{K}^2],\ \ldots\,\}$$

Must check multiple branches to investigate directional stability.

Granular Instability and Softening

Investigating granular instability, $\delta^2 W \overset{?}{<} 0$:

Discrete systems $\longrightarrow$ Search eigenvalues of $[\,\mathbf{K}\,]^{\text{Symmetric}}$

Difficulties:

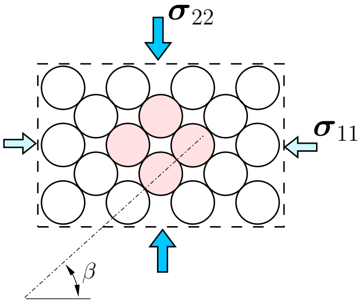
1) Non-symmetric, $[\mathbf{K}] = [\,\mathbf{K}^{\text{Mechanical}}\,] + [\,\mathbf{K}^{\text{Geometric}}\,]$
2) Incrementally non-linear $\Rightarrow$ Multiple stiffness branches,

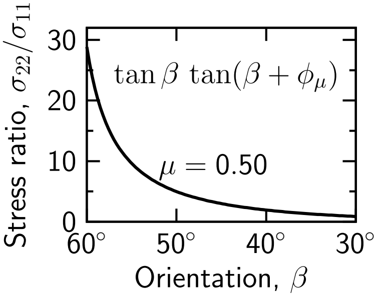
$$[\,\mathbf{K}\,] = \{\, [\,\mathbf{K}^1],\ [\,\mathbf{K}^2],\ \ldots \,\}$$

Must check multiple branches to investigate directional stability.

Granular Instability and Softening — Example

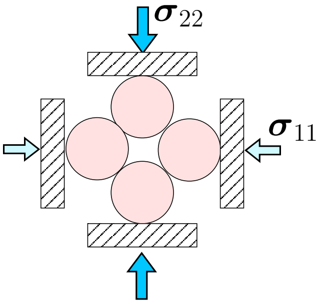
Internal instability during softening





Granular Instability and Softening — Example

Instability of 4 particles



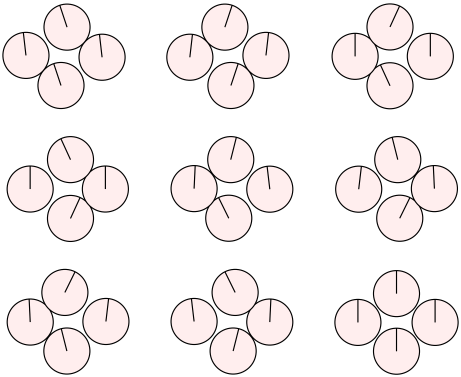
Search for unstable eigenmodes:

$$\lambda < 0 \quad \Rightarrow \quad \delta^2 W < 0$$

Y. Kishino, "Characteristic deformation analysis"

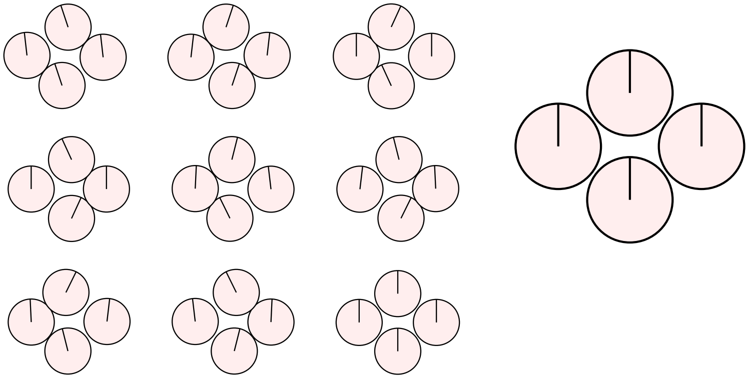
Granular Instability and Softening — Example

9 Unstable eigenmodes, with $\lambda < 0$:



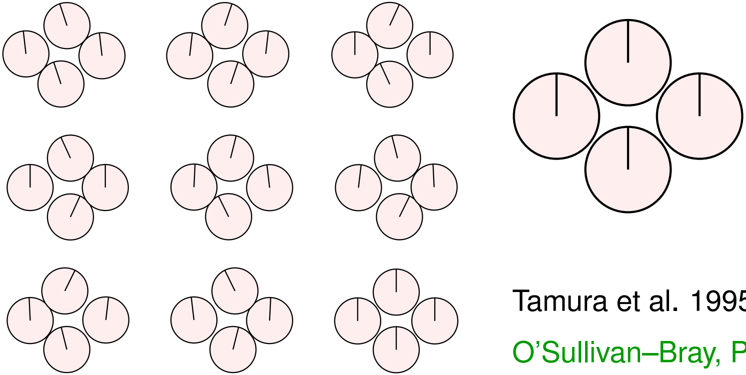
Granular Instability and Softening — Example

9 Unstable eigenmodes, with $\lambda < 0$:



Granular Instability and Softening — Example

9 Unstable eigenmodes, with $\lambda < 0$:



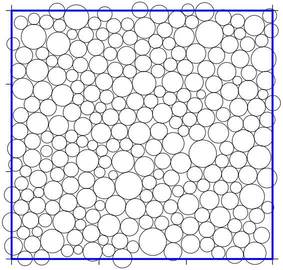
Tamura et al. 1995

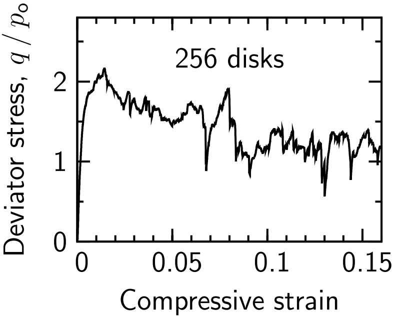
O'Sullivan–Bray, PG2005

Granular Instability and Softening — Another Example

Softening and instability of 256 disks — DEM simulation

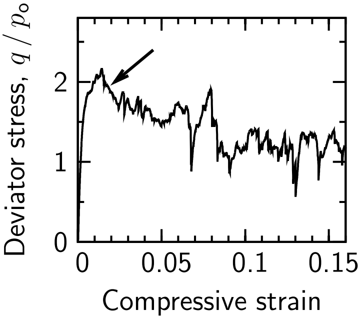
Biaxial compression

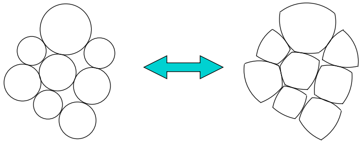




Granular Instability and Softening — Another Example

Softening — effect of contact curvatures

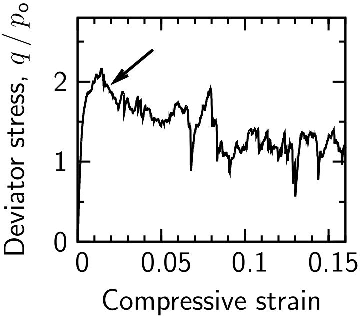


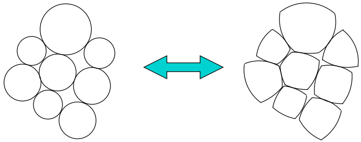


Incremental softening is halted when curvatures "ρ" are increased by 12%.

Granular Instability and Softening — Another Example

Softening — effect of contact curvatures



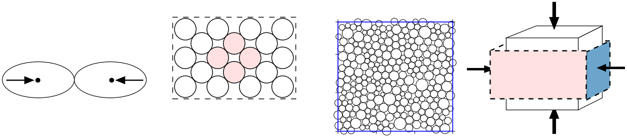


Incremental softening is halted when curvatures "ρ" are increased by 12%.

Granular Instability and Softening

Sources of instability and softening (negative 2nd-order work)

1) Geometric stiffness



2) Mechanical stiffness

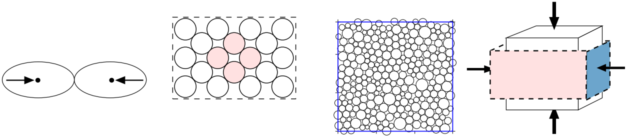
a) Contact friction

b) Particle fracture (Bolton et al., PG2005)

Granular Instability and Softening

Sources of instability and softening (negative 2nd-order work)

1) Geometric stiffness



2) Mechanical stiffness

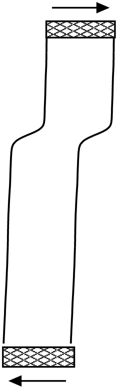
a) Contact friction

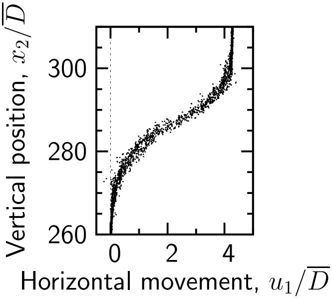
b) Particle fracture (Bolton et al., PG2005)

Outline

Localization — Softening — Instability

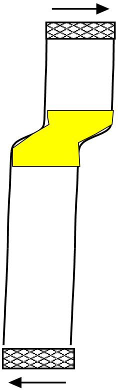
Localization in the shearing of 40,500 disks

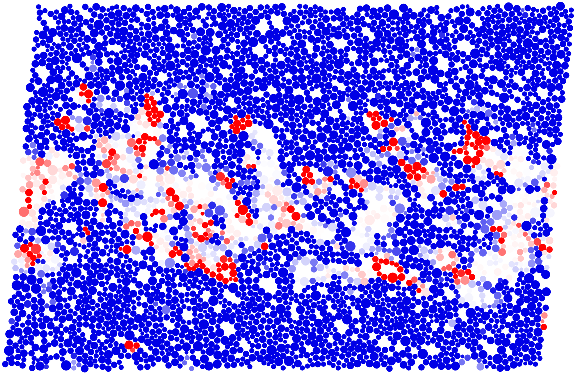




Localization — Softening — Instability

Hardening and softening inside a shear band — $\delta^2 W$





Iwashita & Oda, PG2001

Summary

At large strains . . .

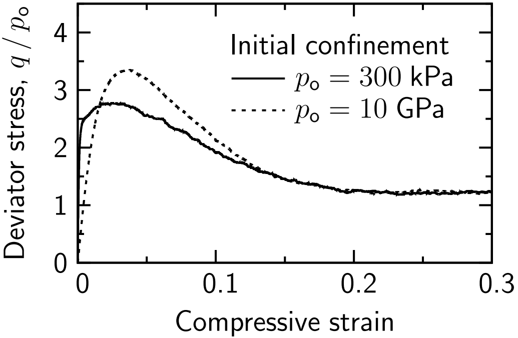
- Granular behavior is dominated by softening, instability, and localization phenomena.
- The study and scaling of these phenomena must account for their mechanical and geometric origins.

Questions

Scaling of Granular Behavior

Effect of confinement pressure on strength

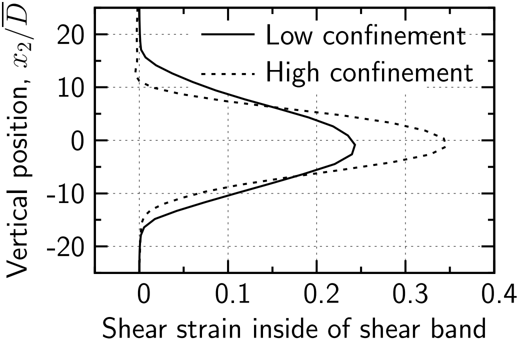
4096 “durable” spheres — DEM simulations



Scaling of Granular Behavior

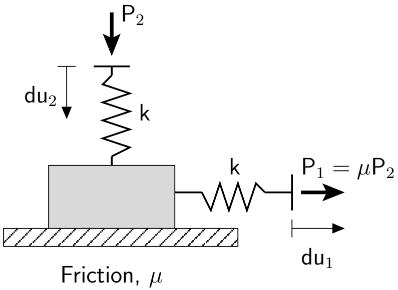
Effect of confinement pressure on shear band thickness

4096 “durable” spheres — DEM simulations



Granular Instability and Softening

Contact friction can produce instability and softening:



Negative 2nd-order work,

$$\delta^2 W = du_1 dP_1 + du_2 dP_2 < 0$$

when $du_2 < 0$ and $du_1 > \frac{1}{\mu}|du_2|$

(Mandel, Bažant)