CHAPTER 4

Development of Beam Equations

Introduction

We begin this chapter by developing the stiffness matrix for the bending of a beam element, the most common of all structural elements as evidenced by its prominence in buildings, bridges, towers, and many other structures. The beam element is considered to be straight and to have constant cross-sectional area. We will first derive the beam element stiffness matrix by using the principles developed for simple beam theory.

We will then present simple examples to illustrate the assemblage of beam element stiffness matrices and the solution of beam problems by the direct stiffness method presented in Chapter 2. The solution of a beam problem illustrates that the degrees of freedom associated with a node are a transverse displacement and a rotation. We will include the nodal shear forces and bending moments and the resulting shear force and bending moment diagrams as part of the total solution.

Next, we will discuss procedures for handling distributed loading, because beams and frames are often subjected to distributed loading as well as concentrated nodal loading. We will follow the discussion with solutions of beams subjected to distributed loading and compare a finite element solution to an exact solution for a beam subjected to a distributed loading.

We will then develop the beam element stiffness matrix for a beam element with a nodal hinge and illustrate the solution of a beam with an internal hinge.

To further acquaint you with the potential energy approach for developing stiffness matrices and equations, we will again develop the beam bending element equations using this approach. We hope to increase your confidence in this approach. It will be used throughout much of this text to develop stiffness matrices and equations for more complex elements, such as two-dimensional (plane) stress, axisymmetric, and three-dimensional stress.

Finally, the Galerkin residual method is applied to derive the beam element equations.

The concepts presented in this chapter are prerequisite to understanding the concepts for frame analysis presented in Chapter 5.

4.1 Beam Stiffness

In this section, we will derive the stiffness matrix for a simple beam element. A **beam** *is a long, slender structural member generally subjected to transverse loading that produces significant bending effects as opposed to twisting or axial effects.* This bending deformation is measured as a transverse displacement and a rotation. Hence, the degrees of freedom considered per node are a transverse displacement and a rotation (as opposed to only an axial displacement for the bar element of Chapter 3).

Consider the beam element shown in Figure 4–1. The beam is of length L with axial local coordinate $\hat{x}$ and transverse local coordinate $\hat{y}$. The local transverse nodal displacements are given by $\hat{d}_{iy}$'s and the rotations by $\hat{\phi}_i$'s. The local nodal forces are given by $\hat{f}_{iy}$'s and the bending moments by $\hat{m}_i$'s as shown. We initially neglect all axial effects.

At all nodes, the following sign conventions are used:

1. Moments are positive in the counterclockwise direction.
2. Rotations are positive in the counterclockwise direction.
3. Forces are positive in the positive $\hat{y}$ direction.
4. Displacements are positive in the positive $\hat{y}$ direction.

Figure 4–2 indicates the sign conventions used in simple beam theory for positive shear forces $\hat{V}$ and bending moments $\hat{m}$.

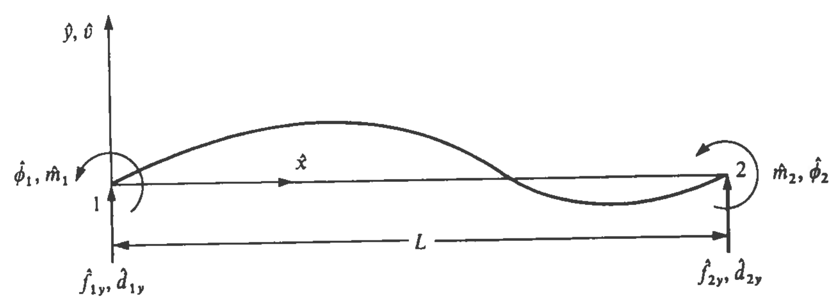


Figure 4–1 Beam element with positive nodal displacements, rotations, forces, and moments

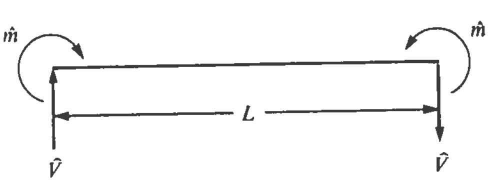


Figure 4–2 Beam theory sign conventions for shear forces and bending moments

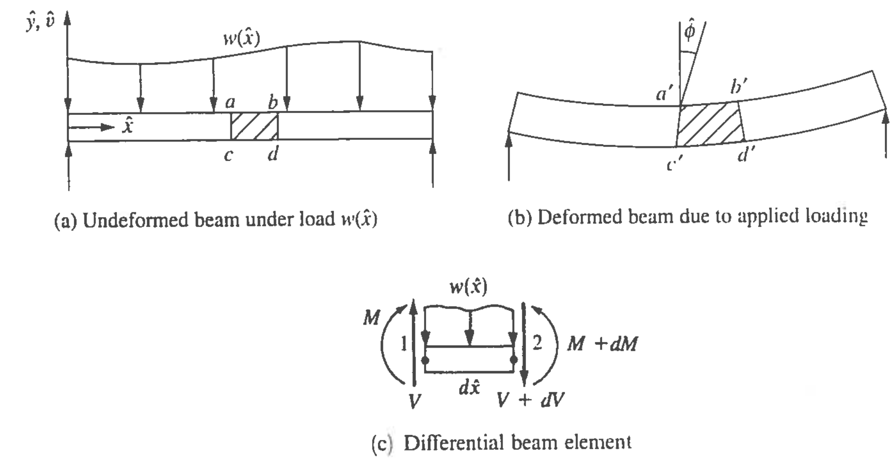


Figure 4–3 Beam under distributed load

Beam Stiffness Matrix Based on Euler-Bernouli Beam Theory (Considering Bending Deformations Only)

The differential equation governing elementary linear-elastic beam behavior [1] (called the Euler-Bernoulli beam as derived by Euler and Bernoulli) is based on plane cross sections perpendicular to the longitudinal centroidal axis of the beam before bending occurs remaining plane and perpendicular to the longitudinal axis after bending occurs. This is illustrated in Figure 4–3, where a plane through vertical line $a-c$ (Figure 4–3(a)) is perpendicular to the longitudinal $\hat{x}$ axis before bending, and this same plane through $a'-c'$ (rotating through angle $\hat{\phi}$ in Figure 4–3(b)) remains perpendicular to the bent $\hat{x}$ axis after bending. This occurs in practice only when a pure couple or constant moment exists in the beam. However it is a reasonable assumption that yields equations that quite accurately predict beam behavior for most practical beams.

The differential equation is derived as follows. Consider the beam shown in Figure 4–3 subjected to a distributed loading $w(\hat{x})$ (force/length). From force and moment equilibrium of a differential element of the beam, shown in Figure 4–3(c), we have

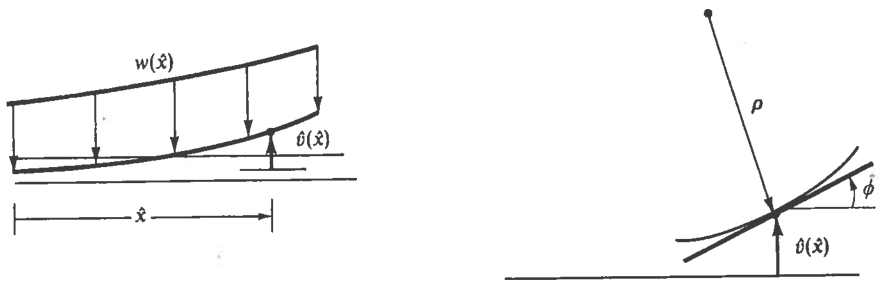
$$\Sigma F_y = 0\text{: } V - (V + dV) - w(\hat{x})\, dx = 0 \tag{4.1.1a}$$

Or, simplifying Eq. (4.1.1a), we obtain

$$-w\, d\hat{x} - dV = 0 \qquad \text{or} \qquad w = -\frac{dV}{d\hat{x}} \tag{4.1.1b}$$

$$\Sigma M_2 = 0\text{: } -V\, dx + dM + w(\hat{x})\, d\hat{x}\left(\frac{d\hat{x}}{2}\right) = 0 \qquad \text{or} \qquad V = \frac{dM}{d\hat{x}} \tag{4.1.1c}$$

The final form of Eq. (4.1.1c), relating the shear force to the bending moment, is obtained by dividing the left equation by $d\hat{x}$ and then taking the limit of the equation as $d\hat{x}$ approaches 0. The $w(\hat{x})$ term then disappears.



(a) Portion of deflected curve of beam

(b) Radius of deflected curve at $\hat{v}(\hat{x})$

Figure 4–4 Deflected curve of beam

Also, the curvature κ of the beam is related to the moment by

$$\kappa = \frac{1}{\rho} = \frac{M}{EI} \tag{4.1.1d}$$

where ρ is the radius of the deflected curve shown in Figure 4–4b, $\hat{v}$ is the transverse displacement function in the $\hat{y}$ direction (see Figure 4–4a), E is the modulus of elasticity, and I is the principal moment of inertia about the $\hat{z}$ axis (where the $\hat{z}$ axis is perpendicular to the $\hat{x}$ and $\hat{y}$ axes).

The curvature for small slopes $\hat{\phi} = d\hat{v}/d\hat{x}$ is given by

$$\kappa = \frac{d^2\hat{v}}{d\hat{x}^2} \tag{4.1.1e}$$

Using Eq. (4.1.1e) in (4.1.1d), we obtain

$$\frac{d^2\hat{v}}{d\hat{x}^2} = \frac{M}{EI} \tag{4.1.1f}$$

Solving Eq. (4.1.1f) for M and substituting this result into (4.1.1c) and (4.1.1b), we obtain

$$\frac{d^2}{d\hat{x}^2}\left(EI\frac{d^2\hat{v}}{d\hat{x}^2}\right) = -w(\hat{x}) \tag{4.1.1g}$$

For constant EI and only nodal forces and moments, Eq. (4.1.1g) becomes

$$EI\frac{d^4\hat{v}}{d\hat{x}^4} = 0 \tag{4.1.1h}$$

We will now follow the steps outlined in Chapter 1 to develop the stiffness matrix and equations for a beam element and then to illustrate complete solutions for beams.

Step 1 Select the Element Type

Represent the beam by labeling nodes at each end and in general by labeling the element number (Figure 4–1).

Step 2 Select a Displacement Function

Assume the transverse displacement variation through the element length to be

$$\hat{v}(\hat{x}) = a_1\hat{x}^3 + a_2\hat{x}^2 + a_3\hat{x} + a_4 \qquad (4.1.2)$$

The complete cubic displacement function Eq. (4.1.2) is appropriate because there are four total degrees of freedom (a transverse displacement $\hat{d}_{iy}$ and a small rotation $\hat{\phi}_i$ at each node). The cubic function also satisfies the basic beam differential equation—further justifying its selection. In addition, the cubic function also satisfies the conditions of displacement and slope continuity at nodes shared by two elements.

Using the same procedure as described in Section 2.2, we express $\hat{v}$ as a function of the nodal degrees of freedom $\hat{d}_{1y}$, $\hat{d}_{2y}$, $\hat{\phi}_1$, and $\hat{\phi}_2$ as follows:

$$\begin{aligned}
\hat{v}(0) &= \hat{d}_{1y} = a_4 \\
\frac{d\hat{v}(0)}{d\hat{x}} &= \hat{\phi}_1 = a_3 \\
\hat{v}(L) &= \hat{d}_{2y} = a_1L^3 + a_2L^2 + a_3L + a_4 \\
\frac{d\hat{v}(L)}{d\hat{x}} &= \hat{\phi}_2 = 3a_1L^2 + 2a_2L + a_3
\end{aligned} \qquad (4.1.3)$$

where $\hat{\phi} = d\hat{v}/d\hat{x}$ for the assumed small rotation $\hat{\phi}$. Solving Eqs. (4.1.3) for a_1 through a_4 in terms of the nodal degrees of freedom and substituting into Eq. (4.1.2), we have

* SEE ATTACHED HAND-NOTES FOR DETAILS *

$$\begin{aligned}
\hat{v} = &\left[\frac{2}{L^3}(\hat{d}_{1y} - \hat{d}_{2y}) + \frac{1}{L^2}(\hat{\phi}_1 + \hat{\phi}_2)\right]\hat{x}^3 \\
&+ \left[-\frac{3}{L^2}(\hat{d}_{1y} - \hat{d}_{2y}) - \frac{1}{L}(2\hat{\phi}_1 + \hat{\phi}_2)\right]\hat{x}^2 + \hat{\phi}_1\hat{x} + \hat{d}_{1y}
\end{aligned} \qquad (4.1.4)$$

In matrix form, we express Eq. (4.1.4) as

$$\hat{v} = [N]\{\hat{d}\} \qquad (4.1.5)$$

where

$$\{\hat{d}\} = \begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix} \qquad (4.1.6a)$$

and where

$$[N] = [N_1 \quad N_2 \quad N_3 \quad N_4] \qquad (4.1.6b)$$

and

$$\begin{aligned}
N_1 &= \frac{1}{L^3}(2\hat{x}^3 - 3\hat{x}^2L + L^3) \qquad & N_2 &= \frac{1}{L^3}(\hat{x}^3L - 2\hat{x}^2L^2 + \hat{x}L^3) \\
N_3 &= \frac{1}{L^3}(-2\hat{x}^3 + 3\hat{x}^2L) \qquad & N_4 &= \frac{1}{L^3}(\hat{x}^3L - \hat{x}^2L^2)
\end{aligned} \qquad (4.1.7)$$

N_1, N_2, N_3, and N_4 are called the **shape functions** for a beam element. These cubic shape (or interpolation) functions are known as *Hermite cubic interpolation (or cubic* *spline) functions.* For the beam element, $N_1 = 1$ when evaluated at node 1 and $N_1 = 0$ when evaluated at node 2. Because N_2 is associated with $\hat{\phi}_1$, we have, from the second of Eqs. (4.1.7), $(dN_2/d\hat{x}) = 1$ when evaluated at node 1. Shape functions N_3 and N_4 have analogous results for node 2.

Step 3 Define the Strain/Displacement and Stress/Strain Relationships

Assume the following axial strain/displacement relationship to be valid:

$$\varepsilon_x(\hat{x}, \hat{y}) = \frac{d\hat{u}}{d\hat{x}} \tag{4.1.8}$$

where $\hat{u}$ is the axial displacement function. From the deformed configuration of the beam shown in Figure 4–5, we relate the axial displacement to the transverse displacement by

$$\hat{u} = -\hat{y}\frac{d\hat{v}}{d\hat{x}} \tag{4.1.9}$$

where we should recall from elementary beam theory [1] the basic assumption that cross sections of the beam (such as cross section $ABCD$) that are planar before bending deformation remain planar after deformation and, in general, rotate through a small angle $(d\hat{v}/d\hat{x})$. Using Eq. (4.1.9) in Eq. (4.1.8), we obtain

$$\varepsilon_x(\hat{x}, \hat{y}) = -\hat{y}\frac{d^2\hat{v}}{d\hat{x}^2} \tag{4.1.10}$$

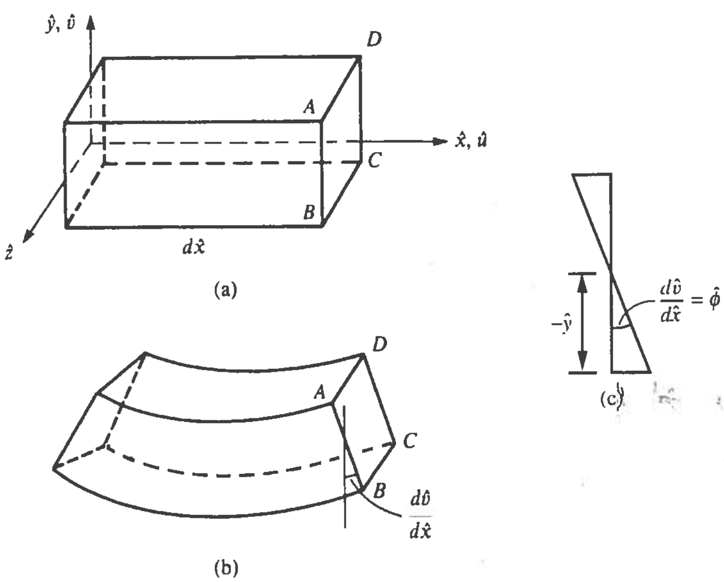


Figure 4–5 Beam segment (a) before deformation and (b) after deformation; (c) angle of rotation of cross section *ABCD*

From elementary beam theory, the bending moment and shear force are related to t transverse displacement function. Because we will use these relationships in the deriv tion of the beam element stiffness matrix, we now present them as

$$\hat{m}(\hat{x}) = EI\frac{d^2\hat{v}}{d\hat{x}^2} \qquad \hat{V} = EI\frac{d^3\hat{v}}{d\hat{x}^3} \tag{4.1.1}$$

Step 4 Derive the Element Stiffness Matrix and Equations

First, derive the element stiffness matrix and equations using a direct equilibriu approach. We now relate the nodal and beam theory sign conventions for shear forc and bending moments (Figures 4–1 and 4–2), along with Eqs. (4.1.4) and (4.1.1 to obtain

$$\begin{aligned}
\hat{f}_{1y} &= \hat{V} = EI\frac{d^3\hat{v}(0)}{d\hat{x}^3} = \frac{EI}{L^3}(12\hat{d}_{1y} + 6L\hat{\phi}_1 - 12\hat{d}_{2y} + 6L\hat{\phi}_2)\\
\hat{m}_1 &= -\hat{m} = -EI\frac{d^2\hat{v}(0)}{d\hat{x}^2} = \frac{EI}{L^3}(6L\hat{d}_{1y} + 4L^2\hat{\phi}_1 - 6L\hat{d}_{2y} + 2L^2\hat{\phi}_2)\\
\hat{f}_{2y} &= -\hat{V} = -EI\frac{d^3\hat{v}(L)}{d\hat{x}^3} = \frac{EI}{L^3}(-12\hat{d}_{1y} - 6L\hat{\phi}_1 + 12\hat{d}_{2y} - 6L\hat{\phi}_2)\\
\hat{m}_2 &= \hat{m} = EI\frac{d^2\hat{v}(L)}{d\hat{x}^2} = \frac{EI}{L^3}(6L\hat{d}_{1y} + 2L^2\hat{\phi}_1 - 6L\hat{d}_{2y} + 4L^2\hat{\phi}_2)
\end{aligned} \tag{4.1.1}$$

where the minus signs in the second and third of Eqs. (4.1.12) are the result of oppo site nodal and beam theory positive bending moment conventions at node 1 an opposite nodal and beam theory positive shear force conventions at node 2 as see by comparing Figures 4–1 and 4–2. Equations (4.1.12) relate the nodal forces to th nodal displacements. In matrix form, Eqs. (4.1.12) become

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix} \tag{4.1.13}$$

where the stiffness matrix is then

$$\hat{\underline{k}} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{4.1.14}$$

Equation (4.1.13) indicates that $\hat{\underline{k}}$ relates transverse forces and bending moments tc transverse displacements and rotations, whereas axial effects have been neglected.

In the beam element stiffness matrix (Eq. (4.1.14) derived in this section, it is assumed that the beam is long and slender; that is, the length, L, to depth, h, dimension ratio of the beam is large. In this case, the deflection due to bending that is predicted by using the stiffness matrix from Eq. (4.1.14) is quite adequate. However, for short, deep beams the transverse shear deformation can be significant and can have the same order of magnitude contribution to the total deformation of the beam. This is seen by the expressions for the bending and shear contributions to the deflection of a beam, where the bending contribution is of order $(L/h)^3$, whereas the shear contribution is only of order (L/h). A general rule for rectangular cross-section beams, is that for a length at least eight times the depth, the transverse shear deflection is less than five percent of the bending deflection [4]. Castigliano's method for finding beam and frame deflections is a convenient way to include the effects of the transverse shear term as shown in [4]. The derivation of the stiffness matrix for a beam including the transverse shear deformation contribution is given in a number of references [5–8]. The inclusion of the shear deformation in beam theory with application to vibration problems was developed by Timoshenko and is known as the Timoshenko beam [9–10].

Beam Stiffness Matrix Based on Timoshenko Beam Theory (Including Transverse Shear Deformation)

The shear deformation beam theory is derived as follows. Instead of plane sections remaining plane after bending occurs as shown previously in Figure 4–5, the shear deformation (deformation due to the shear force V) is now included. Referring to Figure 4–6, we observe a section of a beam of differential length $d\hat{x}$ with the cross section assumed to remain plane but no longer perpendicular to the neutral axis

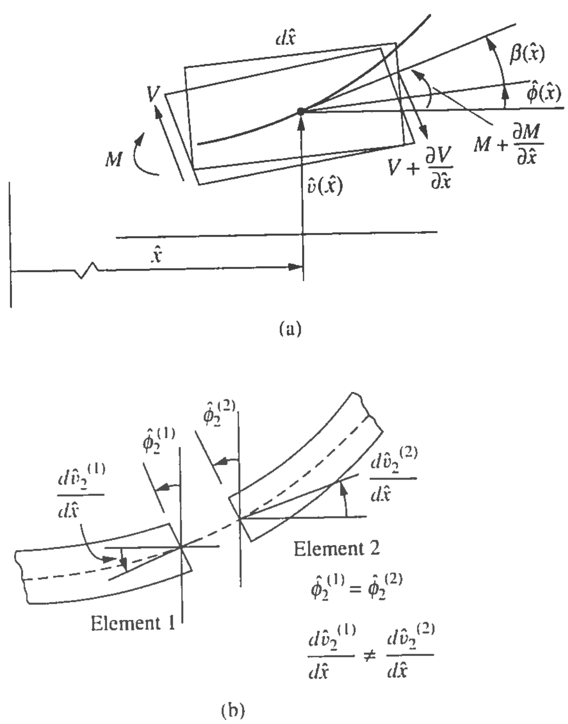


Figure 4–6 (a) Element of Timoshenko beam showing shear deformation. Cross sections are no longer perpendicular to the neutral axis line. (b) Two beam elements meeting at node 2

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$x = 0$ at node 1
$x = L$ at node 2

$$v(x) = a_1 x^3 + a_2 x^2 + a_3 x + a_4$$

$$v(0) = u_1 = a_4$$

$$v(L) = u_2 = a_1 L^3 + a_2 L^2 + a_3 L + a_4$$

$$\phi(x) = v'(x) = 3a_1 x^2 + 2a_2 x + a_3$$

$$\phi(0) = \phi_1 = a_3$$

$$\phi(L) = \phi_2 = 3a_1 L^2 + 2a_2 L + a_3$$

$$a_4 = u_1 \qquad a_3 = \phi_1$$

SOLVE FOR a_2 & a_3

$$3\,v(L) = 3a_1 L^3 + 3a_2 L^2 + 3\phi_1 L + 3u_1 = 3u_2$$

$$L \cdot \phi(L) = 3a_1 L^3 + 2a_2 L^2 + a_3 L = L\phi_2$$

$$3v(L) - L\phi(L) = a_2 L^2 + 2\phi_1 L + 3u_1 = 3u_2 - L\phi_2$$

solving for a_2:

$$a_2 = \frac{-3}{L^2}(u_1 - u_2) - \frac{1}{L}(2\phi_1 + \phi_2)$$

$$\phi(L) = 3a_1 L^2 + 2a_2 L + \phi_1 = \phi_2$$

$$\Rightarrow \quad a_1 = \frac{1}{3L^2}(\phi_2 - \phi_1) - \frac{2a_2}{3L}$$

$$a_1 = \frac{1}{3L^2}(\phi_2 - \phi_1) - \frac{2}{3L}\left(\frac{-3}{L^2}(u_1 - u_2) - \frac{1}{L}(2\phi_1 + \phi_2)\right)$$

$$a_1 = \frac{2}{L^3}(u_1 - u_2) + \frac{1}{L^2}(\phi_1 + \phi_2)$$

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