STRAIN ENERGY DENSITY
(strain energy per unit volume)

For ductile metals and alloys, according to the Maximum Shear Stress failure theory (aka "Tresca") the only factor that affects dislocation slip is the maximum shear stress in the material. This is really a 1-dimensional explanation; a single parameter (maximum shear stress) is the only thing that causes yielding. The Tresca theory does work well in a 3-dimensional world, but none the less, a slight improvement upon Tresca's theory is warranted. Yielding (dislocation slip) is somewhat better explained (i.e. it is better supported by empirical data) by considering strain energy. So here we go….

If we apply a load to a material it will deform. The units of energy are force*distance, so when a load is applied and the material deforms, we are putting energy into the material. This energy is referred to as "strain energy." We prefer to normalize strain energy by unit volume, and when we do so, this is referred to as s*train energy density*. The area under a stress-strain curve is the energy per unit volume; stress*strain has units of force per area such as N/mm^2 which is the same as energy per unit volume N-mm/mm^3. We will be assuming **linear elastic material** only. Most metals and alloys are linear elastic prior to the onset of plastic deformation, so this is generally a valid assumption.

The strain energy is composed of two distinct forms – volume changes and distortion. Normal strains cause a change in volume, shear strain cause distortions (angular changes). The total stain energy is the sum of distortion energy and volume energy:

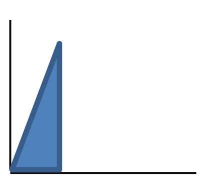
$$U_{total} = U_{distortion} + U_{volume},$$

Where:
- U_{total} = total strain energy
- $U_{distortion}$ = strain energy due to distortion
- U_{volume} = strain energy due to volume change (aka hydrostatic strain energy)

We will develop equations for total strain energy (U_{total}) and volume energy (U_{volume}), and determine the distortion energy (which is really what we are interested in) from:

$$U_{distortion} = U_{total} - U_{volume}$$

Remember, for uniaxial loading, the strain energy per unit volume is the area under the stress-strain curve:



$$U_{total} = ½\, \varepsilon_1 \sigma_1$$

(note, shear stresses do not appear in these equations since we are dealing with principal planes)

For general (3D) loading, the total strain energy is given in terms of principal stresses and strains:

$$U_{total} = \tfrac{1}{2}\,[\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3] \qquad \text{(a)}$$

Using Hooke's law $\varepsilon_1 = [\sigma_1 - \nu(\sigma_2 + \sigma_3)] / E$, $\varepsilon_2 = \ldots$ *etc.* the total strain energy equation (a) can be written in terms of stress only:

$$U_{total} = \{1/2E\}\ \{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\,(\sigma_2\sigma_3 + \sigma_1\sigma_3 + \sigma_1\sigma_2)\} \qquad \text{(b)}$$

Remember that hydrostatic stress causes volume change and that it is invariant (hydrostatic stress is a scalar – it is not directionally dependent – therefore it does not vary depending upon axis orientation). "*Invariant"* means "*does not vary*." The hydrostatic stress can be determined from the average magnitudes of the three principal stresses:

$$\sigma_{hydrostatic} = \sigma_{ave} = (\sigma_1 + \sigma_2 + \sigma_3)/3 \qquad \text{(c)}$$

$\sigma_{hydrostatic}$ is the stress condition that causes volume change. It is invariant. For a moment, let's consider it alone. Let's consider a loading condition that was purely hydrostatic with magnitude of $\sigma_{hydrostatic}$ as calculated in equation (c). If the only stress in this material is $\sigma_{hydrostatic}$ then for this special loading condition the 3 principal stresses would be equal to $\sigma_{hydrostatic}$ ($\sigma_1 = \sigma_2 = \sigma_3 = \sigma_{hydrostatic}$). Equation (b) would become:

$$U = \{(1-2\nu)/6E\}\ \{\sigma_{hyd}^2 + \sigma_{hyd}^2 + \sigma_{hyd}^2 + 2(\sigma_{hyd}\sigma_{hyd} + \sigma_{hyd}\sigma_{hyd} + \sigma_{hyd}\sigma_{hyd})\}$$

$$U = \{3(1-2\nu)/2E\}\ \sigma_{hyd}^2 \qquad \text{(d)}$$

For purely hydrostatic loading condition that we assumed in equation (d), there is no distortion energy ($U_{distortion} = 0$) so $U_{total} = U_{volume}$. But what about our part which may have distortion energy? Regardless of the existence of distortion energy or not, equation (d) – being based on the invariant hydrostatic stress – is the energy due to volume change, U_{volume}:

$$U_{volume} = \{3(1-2\nu)/2E\}\ \sigma_{hyd}^2 \qquad \text{(e)}$$

Substituting equation (c) into (e) gives:

$$U_{volume} = \{3(1-2\nu)/2E\}\ \sigma_{hyd}^2 = \{3(1-2\nu)/2E\}\ \{(\sigma_1 + \sigma_2 + \sigma_3)/3\}^2$$

$$= \{(1-2\nu)/6E\}\ \{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_2\sigma_3 + 2\sigma_1\sigma_3 + 2\sigma_1\sigma_2\} \qquad \text{(f)}$$

To determine the strain energy due to distortion only (not volume change) we subtract equation (f) from equation (b):

$$U_{distortion} = U_{total} - U_{volume} =$$

$$U_{distortion} = \{1/2E\}\ \{\ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\,(\sigma_2\sigma_3 + \sigma_1\,\sigma_3 + \sigma_1\,\sigma_2)\} - \{(1\text{-}2\nu\,)/6E\}\ \{\ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_2\sigma_3 + 2\sigma_1\,\sigma_3 + 2\sigma_1\,\sigma_2\,\} \qquad \text{(g)}$$

Simplifying equation (g) gives:

$$\mathbf{U_{distortion} = \{(1+\nu)/3E\}\ [\ \{\ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}/2\ ]} \qquad \text{(h)}$$

Okay, so what? Remember, the Maximum Shear Stress theory works pretty well in predicting yielding of ductile metals and alloys, but we are trying to improve upon it a bit. Why does Maximum Shear Stress theory work well? Because it is shear stress that causes dislocation slip (aka plastic deformation). What sort of strain do shear stresses produce? They produce shear strains ($\gamma = \tau\,/\,G$); in other words, distortion. What is the equation for maximum shear stress? It is: $\tau_{max} = (\sigma_1 - \sigma_3)\,/\,2$. It is the difference between principal stress divided by 2. What do we see in equation (h)? The differences between all principal stress divided by 2. Equation (h) combines the maximum shear stress in each of the 3 principal planes into a single equation. It should not be surprising that "distortion strain energy" is related to maximum shear stress. Shear stress cause shear strain, which is distortion.

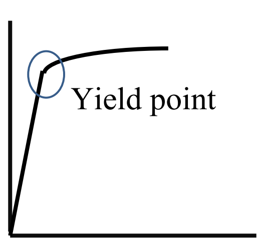
The *Distortion Energy* failure theory (which we will discuss next) is a bit more mathematically sophisticated than the *Maximum Shear Stress* failure theory, but is really very similar. Rather than considering only *the* maximum shear stress at a point, it combines the each of maximum shear stresses at a point on the 3 principal planes. These two theories give very similar results, but Distortion Energy does match empirical data better.

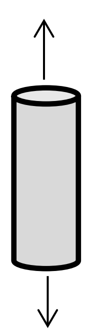
DISTORTION ENERGY FAILURE THEORY (VON MISES FAILURE THEORY):

> ***Yielding is predicted to occur when the distortion energy in a part equals or exceeds the distortion energy in a uniaxial loaded tensile bar at the onset of yielding.***

Note on nomenclature: in machine design, stress related material properties are expressed as S and σ is used to express stresses. Yield strength is expresses as S_{ys} rather than as σ_{ys} (exact same thing, just different nomenclature).

For uniaxial tensile loading (as is used to create a stress-strain curve), $\sigma_2 = \sigma_3 = 0$, and at the onset of yielding, $F/A = \sigma_1 = S_{ys}$ (at onset of yielding only).





Therefore, for uniaxial loading at the onset of yielding (the stress shown on the stress-strain curve that we call "yield strength") we substituting S_{ys} for σ_1 and $\sigma_2 = \sigma_3 = 0$ into equation (h):

$$U_{distortion} = \{(1+ \nu)/3E\} S_{ys}^2 \qquad \text{(i)}$$

The *Distortion Energy Theory* states that when the distortion energy in a material equals or exceeds the distortion energy present at the onset of yielding in uniaxial loading tensile test for that material, the part will experience plastic deformation (i.e. it will yield):

$$U_{distortion,\ part} \geq U_{distortion,\ uniaxial\ test} \ \text{ yielding occurs} \qquad \text{(j)}$$

Equating the distortion energy in general 3-dimensional stress condition (equation (h)) and distortion energy in simple uniaxial loading (equation (i)); from equations (h) and (i) into equation (j):

$$\{(1+ \nu)/3E\} \ \{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}/2 \ \geq \ \{(1+ \nu)/3E\} S_{ys}^2 \qquad \text{(k)}$$

Simplifying equation (k) gives the von Mises failure criterion:

$$\textbf{If } \{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\} / 2 \ \geq \ S_{ys}^2 \ \textbf{ then yielding will occur} \qquad \text{(l)}$$

We define the von Mises stress, a.k.a. effective stress, as:

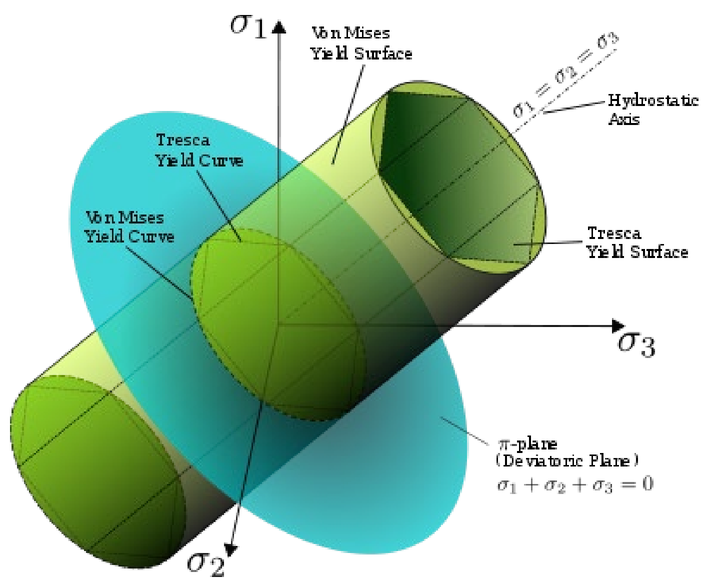
$$\sigma' = \sigma_{effective} = \sigma_{eff} = [\{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}/2]^{1/2} \qquad \text{(m)}$$

Yielding occurs when the von Mises stress (aka effective stress) in a part becomes greater than the yield strength:

$$\boldsymbol{\sigma' = \sigma_{effective} > S_{ys}} \qquad \text{(n)}$$

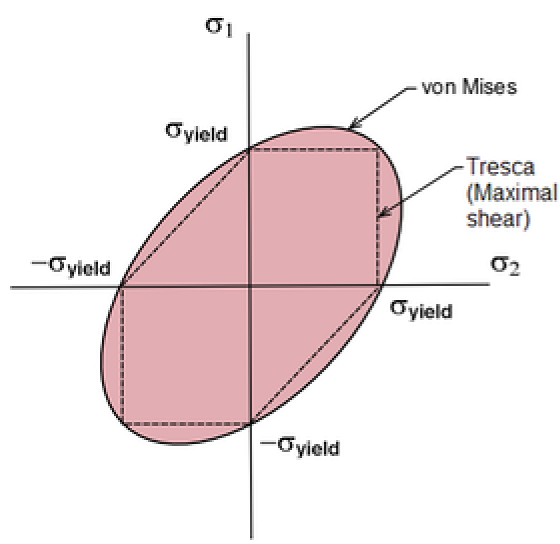
Equations (l) and (n) are the same equation. They are both the *von Mises theory (distortion energy theory)* in mathematical form.

What have we done? Principal stresses lie along specific orthogonal directions and therefore are not scalar entities. However, what we have done is to use their magnitude in a scalar equation to express the distortion energy. We have also combined the principal stresses in another equation to define the so-called effective stress (aka von Mises stress). The effective stress is a scalar quantity (no direction associated with it) that is related to the yield strength (also a scalar quantity) to predict the complex phenomena of yielding. It was a long way to go, but are able to relate complex loading conditions to a simple material property in order to predict yielding (failure) of our complex loaded part! That's impressive!



von Mises (Distortion Energy theory) and Tresca (Max Shear Stress theory) yield surfaces.

For plane stress (let σ_3=0 and $\sigma_1 > \sigma_2$) the circle passing through the cylindrical shape (von Mises) becomes an ellipse and the hexagon (Tresca) becomes an elongated hexagon:



von Mises (Distortion Energy theory) and Tresca (Max Shear Stress theory) failure surfaces for plane stress (let σ_3=0 and $\sigma_1 > \sigma_2$).

HISTORICAL NOTES
(Wikipedia): Henri Tresca (October 12, 1814 – June 21, 1885) was a French mechanical engineer, and a professor at the Conservatoire National des Arts et Métiers in Paris.

Maximum Shear Stress theory was developed by Tresca in 1868.

Tresca was also among the designers of the prototype meter bar (shown below) that served as the first standard of length for the metric system. This was a very challenging “strength of materials” problem. The bar is carefully designed to minimally “sag” when supported at its ends, and it is designed to easily be measured at its neutral axis (no change in length at NA).

History of **Distortion Energy Failure Theory**, aka von Mises, aka von Mises-Hencky, aka Huber-Hencky-von Mises (http://www.continuummechanics.org/cm/vonmisesstress.html)

The defining equation for the von Mises stress was first proposed by Huber [1] in 1904, but apparently received little attention until von Mises [2] proposed it again in 1913. However, Huber and von Mises' definition was little more than a math equation without physical interpretation until 1924 when Hencky [3] recognized that it is actually related to deviatoric strain energy.

In 1931, Taylor and Quinney [4] published results of tests on copper, aluminum, and mild steel demonstrating that the von Mises stress is a more accurate predictor of the onset of metal yielding than the maximum shear stress criterion, which had been proposed by Tresca [5] in 1864 and was the best predictor of metal yielding to date. Today, the von Mises stress is sometimes referred to as the Huber-Mises stress in recognition of Huber's contribution to its development. It is also called Mises effective stress and simply effective stress.
Huber, M.T. (1904) Czasopismo Techniczne, Lemberg, Austria, Vol. 22, pp. 181.
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2. Hencky, H.Z. (1924) "Zur Theorie Plasticher Deformationen und der Hierdurch im Material Hervorgerufenen Nachspannungen," Z. Angerw. Math. Mech., Vol. 4, pp. 323.
3. Taylor, G.I., Quinney, H. (1931) "The Plastic Distortion of Metals," Phil. Trans. R. Soc., London, Vol. A230, pp. 323.
4. Tresca, H. (1864) "Sur l'Ecoulement des Corps Solides Soumis a de Fortes Pressions," C. R. Acad. Sci., Paris, Vol. 59, pp. 754.
5. Dowling, N.E. (1993) Mechanical Behavior of Materials, Prentice Hall.