ME304 Finite Element Analysis
Stress, Strain, Hooke's Law

Compatibility
One of the mathematical requirements for continuum mechanics is "compatibility". **Compatibility** simply means there are no "holes or gaps" in the material. The for plane strain, three strains are defined by only two displacements; therefore, the strains are not independent of each other! For **plane strain**, the **compatibility condition** is:

$$\partial^2\varepsilon_{xx}/\partial y^2 + \partial^2\varepsilon_{yy}/\partial x^2 - \partial^2\varepsilon_{xy}/\partial x\partial y = 0$$

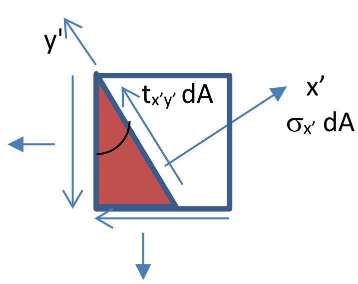
> *To gain further insight into the meaning of compatibility, imagine an elastic body subdivided into a number of small cubic elements prior to deformation. These cubes may, upon loading, be deformed into a system of parallelepipeds. The deformed system will, in general, be impossible to arrange in such a way as to compose a continuous body unless the components of strain satisfy the equations of compatibility.* Ugural and Fenster, "Advanced Strengths and Applied Elasticity", Elsevier, pg. 40

Matrices!
Scalars are **zero-th order tensors** – a magnitude.
Vectors are **first order tensors** (magnitude and direction which can be represented by a column or row matrix – i.e. one dimensional matrix).
Stress and strain are **second order tensors** ("2 dimensional" matrix – square)

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Where σ_{xx} is the normal stress (acting on the x-plane) in the x-direction (this is also commonly labeled as σ_x). σ_{xy} is the shear stress on the x-plane in the y-direction (also commonly labeled as τ_{xy}), etc. For isotropic materials, the matric is symmetric ($\sigma_{xy} = \sigma_{yx}$).

Transformation Equations for <u>Plane Stress</u> (see *strength of materials* text for details):



$$\sigma_{x'} = ½ (\sigma_x + \sigma_y) + \{ ½ (\sigma_x - \sigma_y) \cos(2\theta) + \tau_{xy} \sin(2\theta) \}$$
$$\sigma_{y'} = ½ (\sigma_x + \sigma_y) - \{ ½ (\sigma_x - \sigma_y) \cos(2\theta) + \tau_{xy} \sin(2\theta) \}$$ **Equations 1**

$\tau_{x'y'} = ½ (\sigma_x – \sigma_y) \sin(2\theta) + \tau_{xy} \cos(2\theta)$

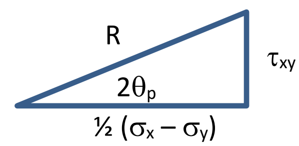
We can see that the average normal stress is constant regardless of orientation:

$\sigma_{ave} = ½ (\sigma_{x'} + \sigma_{y'}) = ½ (\sigma_x + \sigma_y)$ therefore, the average normal stress is said to be *invariant.*

Let's consider the second portion of the transformation equations for normal stresses:

$\{ ½ (\sigma_x – \sigma_y) \cos(2\theta) + \tau_{xy} \sin(2\theta) \}$

If we let $R = \{ [½ (\sigma_x – \sigma_y)]^2 + [\tau_{xy}]^2 \}^{1/2}$ we can create a special triangle – one where the angle is specifically defined as $2\theta_p$



$2\theta_p = \tan^{-1}(2\tau_{xy}/(\sigma_x – \sigma_y)$

$R \cos(2\theta_p) = ½ (\sigma_x – \sigma_y)$

$R \sin(2\theta_p) = \tau_{xy}$

Substituting these into Equations 1 and setting $\theta = \theta_p$ gives us equations that can be used to create a circle (Mohr's circle):

$\sigma_{x'} = \sigma_{ave} + R \{\cos(2\theta_p) + \sin(2\theta_p) \}$

$\sigma_{y'} = \sigma_{ave} – R \{\cos(2_p\theta) + \sin(2_p\theta) \}$ **Equations 2**

Three-Dimensional Stress-Strain (non-plane stress and non-plane strain)

If we know the three principal stresses, 3D Mohr's circle is trivial. However, determining the principal stresses from a general 3D loading condition is more complex. Stress is a square matrix (3X3) and the eigenvalues are the principal stresses and the eigenvectors are the principal directions (see any linear algebra text for further discussion of eigenvalues). The principal stresses are invariant (see comment below for what this means).

$\det[\boldsymbol{\sigma}_{ij} - \mathbf{I}\, \sigma_p] = 0$ ($\boldsymbol{\sigma}_{ij}$ is a 3X3 matrix, $\mathbf{I}$ is the 3X3 identity matrix, σ_p is a scalar (eigenvalue))

expanding this gives:

$\sigma_p^3 – I_1\sigma_p^2 + I_2\sigma_p – I_3 = 0$ (the 3 roots of this equation are the 3 principal stresses)

Where:

$I_1 = \sigma_x + \sigma_y + \sigma_z$

$I_2 = \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z – \tau_{xy}^2 – \tau_{xz}^2 – \tau_{yz}^2$

$I_3 = \det[\boldsymbol{\sigma}]$

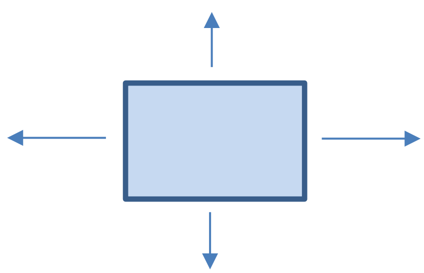
REF: Ugural and Fenster, "Advanced Strengths and Applied Elasticity", Elsevier, pg. 20

The Is are invariant: *No matter what coordinate transformation you apply to the stress tensor, its principal stress must be the same three values, and the principal directions must be the same three vectors. And the only way for this to happen in the above equation is for the equation itself to always be the same, no matter the transformation. This means that the combinations of stress components, which serve as coefficients of the eigenvalues (σ_p), must be invariant under coordinate transformations. Their values must not change. And that's why they are called invariants.*
REF: http://www.continuummechanics.org/principalstress.html

Hooke's Law

Hooke's law for 1-dimensional loading (aka, uni-axial loading) is very familiar: $\boldsymbol{\sigma = E\,\varepsilon}$ **or** $\boldsymbol{\varepsilon = E^{-1}\sigma}$

Hooke's law for bi-directional (2-dimensional) loading (**plane stress, σ_z=0**) is relatively easily developed from 1D if we understand Poisson's ratio.



If σ_y was zero, the strain in the x-direction due to stress in the x-direction alone would be:

$\varepsilon_x = (1/E)\,(\sigma_x)$ …this is the strain in the x-direction due to a stress in the **x**-direction.

If σ_x was zero, the strain in the x-direction due to stress in the y-direction alone would be caused by "Poisson's effect." By definition of Poisson's ratio (valid for uniaxial loading): $\nu = -\varepsilon_{trans}/\varepsilon_{axial}$; therefor:

$\varepsilon_x = -\,\nu\,\varepsilon_y = -\nu\,(1/E)\,(\sigma_y)$ …this is the strain in the x-direction due to a stress in the **y**-direction.

Due to linearity, we can add these. The strain in the x-direction due to the stresses in both the x and y directions:

$\varepsilon_x = (1/E)\,\{\,\sigma_x - \nu\,\sigma_y\,\}$

Similarly, we can determine the strain in the y-direction due to stresses in both x and y directions:

$\varepsilon_y = (1/E)\,\{\,\sigma_y - \nu\,\sigma_x\,\}$

The stresses in the x-y plane will result in strain in the z-direction:

$\varepsilon_z = (-\nu/E)\,\{\,\sigma_x + \sigma_y\,\}$

Notice, for plane stress (σ_z =0) if σ_x = -σ_y, then ε_z=0. If there are no applied shear stresses in the z-direction (which would produce shear strains in the z-direction), then both plane stress and plane strain exist. There is one common loading condition where this is true (hint, draw Mohr's circle for torsion loading…that will show you that σ_x = -σ_y and at least on on the surface of the shaft, it is plane stress…and plane strain).

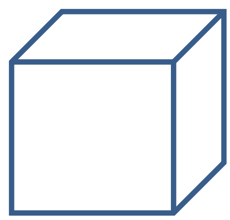
Simultaneously solving the above equations in terms of stress (σ_x, σ_y) results in the following. Note that the strain in the z-direction (the direction of zero stress) has no influence on the stresses in any direction.

$$\sigma_x = \{E / (1 - \nu^2)\} \{\varepsilon_x + \nu\varepsilon_y\}$$

$$\sigma_y = \{E / (1 - \nu^2)\} \{\varepsilon_y + \nu\varepsilon_x\}$$

$\sigma_z = 0$ (due to plane stress)

Three dimensional (**<u>non-plane stress</u>**) Hooke's law.



Hooke's law in 3-dimensions, given the stresses the strains are determined:

$$\varepsilon_x = (1/E) \{ \sigma_x - \nu (\sigma_y + \sigma_z) \}$$

$$\varepsilon_y = (1/E) \{ \sigma_y - \nu (\sigma_x + \sigma_z) \}$$

$$\varepsilon_z = (1/E) \{ \sigma_z - \nu (\sigma_x + \sigma_y) \}$$

Solving these in terms of stress (σ_x, σ_y, σ_z) results in following:

$$\sigma_x = \{E / [(1+\nu)(1-2\nu)]\} \{ (1-\nu)\varepsilon_x + \nu (\varepsilon_y + \varepsilon_z) \}$$

$$\sigma_y = \{E / [(1+\nu)(1-2\nu)]\} \{ (1-\nu)\varepsilon_y + \nu (\varepsilon_x + \varepsilon_z) \}$$

$$\sigma_z = \{E / [(1+\nu)(1-2\nu)]\} \{ (1-\nu)\varepsilon_z + \nu (\varepsilon_x + \varepsilon_y) \}$$

What about shear stress and shear strain? The shear components are independent of the normal components – shear does not affect normal, and normal does not affect shear.

$$\gamma_{xy} = (1/G)\, \tau_{xy}$$

Note, τ_{xy} is different nomenclature, but exactly the same thing as σ_{xy}. However, as discussed previously, $\gamma_{xy} = 2\ \varepsilon_{xy}$ (both are shear strain, but based on how they are defined they differ by a factor of 2).

And $\tau_{xy} = (G)\ \gamma_{xy}$

The same relationship exists for all components of shear stress and shear strain ($\gamma_{xz} = (1/G)\ \tau_{xz}$, $\tau_{xz} = (G)\ \gamma_{xz}$, etc.)

Hooke's law expressed as matrices:

$\sigma = E\ \varepsilon$ or $\varepsilon = E^{-1}\ \sigma$

If stress (σ) and strain (ε) are expressed as 3X3 matrices, the stiffness matrix ("**E**" or "**E^{-1}**") would also need to also be a 3X3. Yet based on the above Hooke's law equations, it can quickly be seen that this won't work! Why not?

Both stress and strain are second order tensors which mean that we can legitimately express them as "2-dimensional" (square 3X3) matrices. However, the **elastic stiffness matrix (E) is a 4th order tensor**; therefore, it would require a "4-dimensional" matrix --- we need "4-dimensional" paper and a "4-dimensional" mind to properly express it. So matrices, in this form, do not work – we need to use tensor form of matrices. As a tensor, we express both stress and strain as 6X1 matrices (even though stress and strain are NOT vectors!), and "stiffness" as a 6X6 matrix.

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}$$