ME 328 – Machine Design
Vibration handout (vibrations is not covered in text)

The following are two good textbooks for vibrations (any edition). There are numerous other texts of equal quality.

M. L. James, et al., *Vibration of Mechanical and Structural Systems,* Harper-Collins College Publishers.

R. Vierck, *Vibration Analysis*, Harper and Row.

The following is an outline of vibration subjects covered in ME328. Only single degree of freedom will be investigated:

- Basic concepts of vibrations and waves
  - What are the basic features of harmonic motion?
- Analysis of simple harmonic motion of undamped free vibration
  - What is the natural frequency?
- Basic behavior of viscous damped free vibration.
  - What is the qualitative behavior of a damped system?
- Analysis of forced vibration without dampening.
  - What is the vibration amplitude if an undamped system is excited with force input?
- Analysis of forced vibration with viscous damping.
  - What is the vibration amplitude if a damped system is excited with force input?
- Isolation and transmission of vibration forces of forced vibration with viscous damping.
  - How much force is transmitted from the vibrating system to the surrounding support structure?

VIBRATION BASICS

In this section we will study basic features of harmonic motion.

Why is vibration worth studying? What effects does it have (good and bad)?

What is meant by “degree of freedom” (DOF)?

Sketch a 1 DOF system:

How many degrees of freedom does a rigid body have, and what are they?

Sketch a 2 DOF system:

How many degrees of freedom does a real structure have, say for example a simple cantilever beam or a guitar string?

What is meant by “mode”?

How many modes were apparent in the Tacoma Narrows Bridge? Describe them.

What is meant by “fundamental frequency”?

What are “overtones”?

Waves:
Vibration is a oscillating motion, therefore it is essential to define basic concepts of sinusoidal waves.

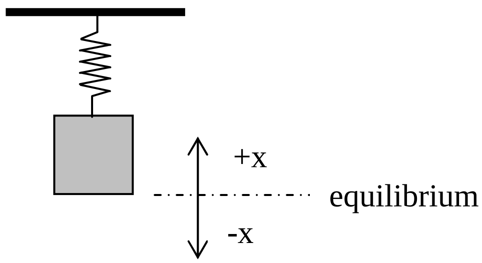
$x(t) = A \sin(\omega t + \phi)$
$x(t)$ = displacement or amplitude as a function of time (meters)
t = time (seconds)
ω = angular frequency (radians per second)
ϕ = phase angle
A = peak amplitude (meters)

τ = period, how much time per cycle
f = frequency, cycles per second (1 Hz = 1 cycle/sec)
$f = 1/\tau$
$\omega = 2\pi f$

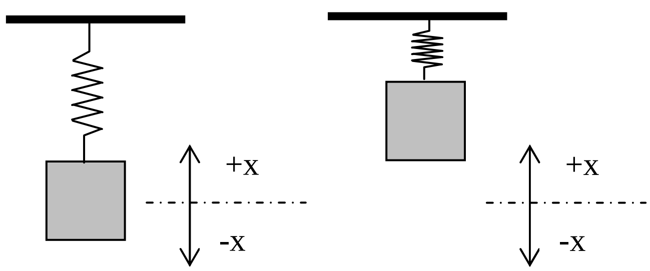
SIMPLE HARMONIC MOTION OF UNDAMPED FREE VIBRATION

In this section, the main objective is to determine natural frequency of a system.

Simple mass-spring system at equilibrium (not moving):



If a system is displaced from equilibrium it will move to seek out equilibrium. If there is no damping, the system will continually convert energy back and forth between kinetic energy of the moving mass and elastic energy of the spring.



Assumptions we will make for analysis:
1 Degree of freedom, which requires (why, how will system behave differently if assumptions are not valid?):

- *Rigid support*
- *Massless, linear elastic spring*
- *Displacement of mass is in one direction only*
- *Mass is rigid*
- *No losses (no damping, etc.)*

k = spring stiffness (lb/in, N/m, etc.)
ω_n = natural angular frequency = $2\pi/\tau_n$ (radians/second)
f_n = natural frequency = $1/\tau_n$ (cycles/second, Hz)
What is natural frequency?

How will the mass affect natural frequency? Will increased mass increase the natural frequency or decrease it?

How will spring stiffness affect natural frequency? Will increased stiffness increase the natural frequency or decrease it?

Sketch FBD & kinetic diagram:

Solutions to equations of motion:

$x = A \sin(\omega_n t)$	displacement as a function of time
$\dot{x} = A\,\omega_n \cos(\omega_n t)$	velocity as a function of time
$\ddot{x} = -A\,\omega_n^2 \sin(\omega_n t)$	acceleration as a function of time

Equilibrium equation:

inertial forces = - elastic forces (momentum opposes the elastic restoring force)

$$m\ddot{x} = -kx \quad = \quad m(-A\,\omega_n^2 \sin(\omega_n t)) = -k\,(A \sin(\omega_n t))$$

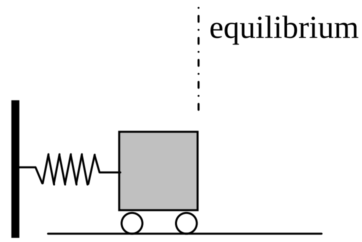
Solving the equation gives:

$$\omega_n^2 = k/m; \quad \omega_n = \sqrt{k/m} \qquad \text{Eq. 1}$$

Does this make sense? Did you expect an increase in stiffness to increase the natural frequency? Did you expect an increase in mass to decrease the natural frequency?

If the mass-spring system was rotated and placed on frictionless rollers such that gravity is perpendicular to the direction of motion,

a) How would natural frequency change?
b) How would amplitude (A) of oscillation change?
c) How would the equilibrium position change?



Example

Given: An elevator car with mass of 4000kg

Suspended by a steel cable, 20m long, 6cm^2 cross-section

Find: Natural frequency of the system

Assume:

Sketch:

Solution:

Example

Given: An aluminum cantilever beam with cross section of 10cm by 10cm

A 100kg reciprocating single piston air compressor is placed at the end of the beam

The compressor spins at 3600RPM

Find: Beam length such that $\omega_n = \omega_f$

Assumptions:

Sketch:

Solution:

VISCOUS DAMPED FREE VIBRATION

(TRANSIENT CONDITION)

The main objective of this section is to understanding damping. We will focus on qualitative behavior, not quantitative. This is the only condition we will look at where the motion is transient (non-steady state).

What typically creates viscous damping?

What is the damping force proportional to for viscous damping?

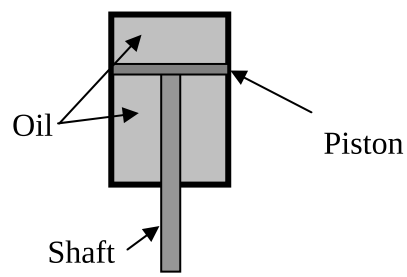
Are there other types of damping?

What is the force proportional to for friction damping?

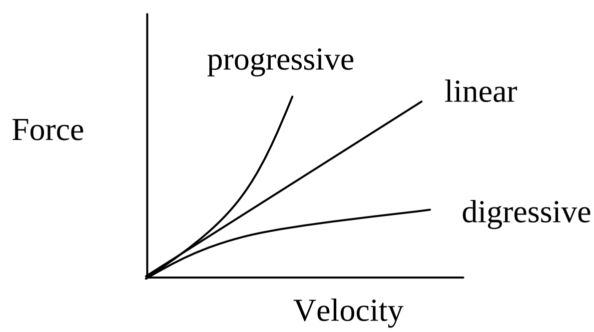
Will we study other forms of damping in ME328? No.

Tidbits of shock absorber history:

- 1901 – first patent for hydraulic shock absorber
- 1925 – hydraulic shocks used on automobiles

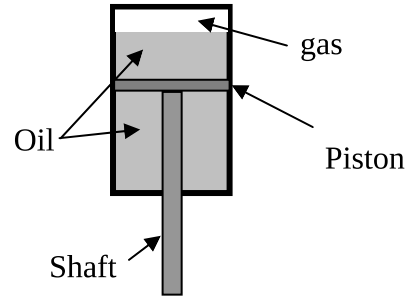


Holes in piston allow oil to flow from one side to the other (viscous forces are produced). Various relationships between velocity and force can be created by varying design:



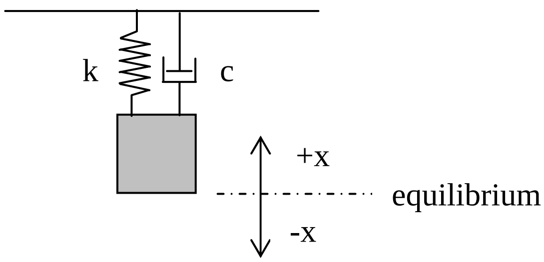
Progressive and digressive shock absorbers can be created by using shims (flaps) to partially cover the passage holes in the piston giving a non-linear response. This can also be used to create a different response depending on the direction of travel (up verses down for example). <u>We will only consider linear response</u> (force is directly proportional to velocity).

Gas shock absorbers use gas to pressurize the oil (typically nitrogen). What two effects does this have?



The system we will study consists of a mass, spring, and dashpot (shock absorber). The dashpot is viscous (force is proportional to velocity).

Sketch FBD & kinetic diagram



k = spring constant (as before)

c = damping constant

Equilibrium equation:

acceleration force = - elastic force - damping force

$m\ddot{x} = -kx - c\dot{x}$ dividing by mass gives: $\ddot{x} + c/m\,\dot{x} + k/m\,x = 0$

The critical damping constant, c_c is:

$c_c = 2m\omega_n$ Eq. 2

The damping factor, ζ, is:

$\zeta = c/c_c$ Eq. 3

What is critical damping? ("*critical*" is not the same as "*very important*")

What are the units of the damping constant, c?

If a mass, spring, dashpot system is displaced from equilibrium and released (zero initial velocity), on the same graph sketch the response for:

a) $\zeta < 1$ (under-damped) (ζ = Zeta)

b) $\zeta = 1$ (critically damped)

c) $\zeta > 1$ (over-damped)

If a critically damped system is displaced from equilibrium and released with the following initial velocities, sketch the response

a) $\dot{x} > 0$ (velocity away from equilibrium position)

b) $\dot{x} = 0$ (released without initial velocity)

c) $\dot{x} < 0$ (velocity towards equilibrium position)

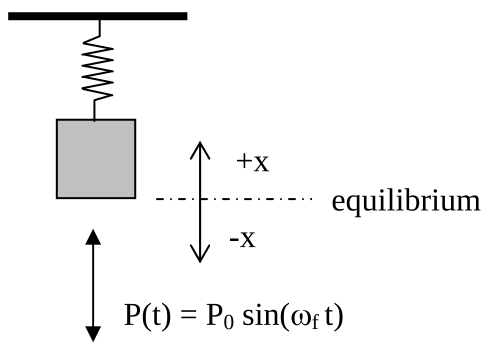
ANALYSIS OF FORCED VIBRATION WITHOUT DAMPENING

(STEADY STATE CONDITION)

In this section we will study the effect of harmonic forced input (an external force applied sinusoidally). We are primarily interested in determining the maximum displacement of the system.

What are some potential sources of harmonic forces?

Sketch FBD & kinetic diagram



P(t) is a harmonic force input. Question: what is ω_f? How is it different than ω_n?

P is the harmonic force. It produces a "push" and a "pull" in an oscillatory manner with an angular frequency of ω_f. ω_f is sometimes referred to as the forcing frequency.

Equilibrium equation:

$$m\ddot{x} = -kx + P(t) = -kx + P_0 \sin(\omega_f t)$$

Solving:

$$x(t) = \frac{P_0}{k - m\omega_f^2}\sin(\omega_f t) = \frac{P_0/k}{1 - \omega_f^2 / (k/m)}\sin(\omega_f t) = \frac{P_0/k}{1 - \left(\omega_f / \omega_n\right)^2}\sin(\omega_f t) \qquad \text{Eq. 4}$$

Let $X_0 = P_0/k$ $\quad$ X_0 is the static displacement due to the ***static*** force P_0. $\qquad$ Eq. 5

Let $r = \omega_f / \omega_n$ $\quad$ r is known as the frequency ratio. $\qquad$ Eq. 6

In this class, we are NOT concerned with x(t), but we are concerned with amplitudes.

Substituting X_0 and r into the previous equation for $x(t)$ gives:

$$x(t) = \frac{X_0}{1-r^2} \sin(\omega_f t) \qquad \text{Eq. 7}$$

Now let $$X = \frac{X_0}{\left|1-r^2\right|} \qquad \text{Eq. 8}$$

X is the peak amplitude of displacement it is ***<u>not</u>*** a function of time. Finally,

$$x(t) = X \sin(\omega_f t) \qquad \text{Eq. 9}$$

Three conditions may exist;

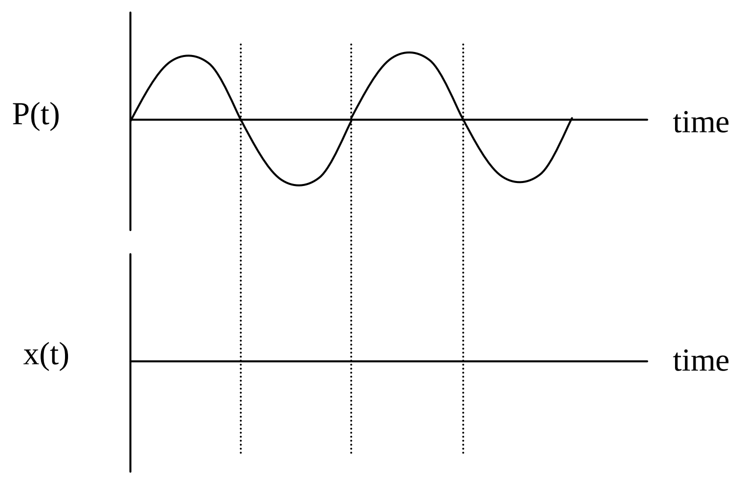
r < 1 (forcing frequency is less than the natural frequency; $\omega_f < \omega_n$)

r > 1 (forcing frequency is greater than the natural frequency; $\omega_f > \omega_n$)

r = 1 (forcing frequency is equal to the natural frequency; $\omega_f = \omega_n$)

If r < 1

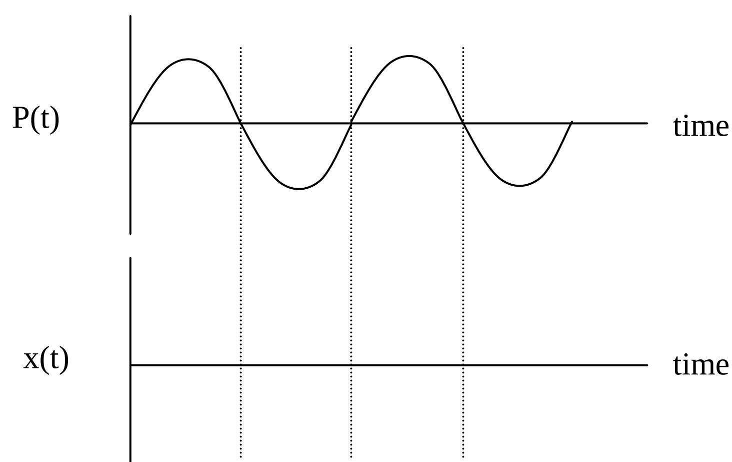
sketch x(t)



If r > 1

sketch x(t)

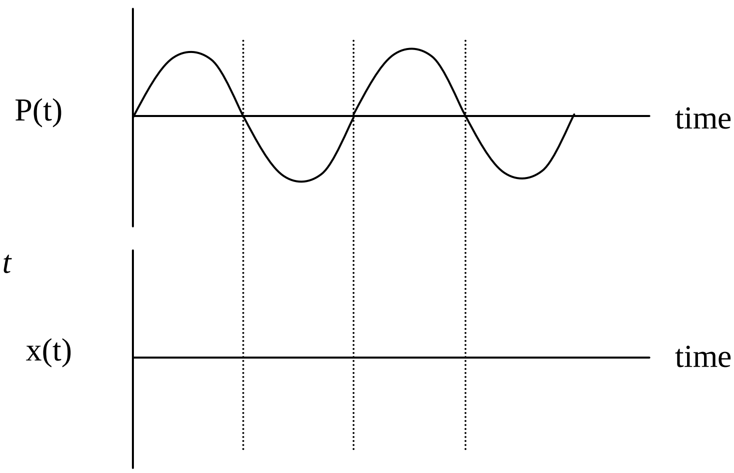
How is the phase different than for r < 1?



If r = 1

Then:

$$x(t) = -\frac{X_o \omega_f t}{2} \cos \omega_f t$$



An important question to answer is: What is the amplitude of the actual vibration?

The magnification factor, MF, is the ratio of the actual vibration amplitude (X) normalized by the displacement (X_0). In all cases MF is a function of the frequency ratio, r. Remember r = $\frac{\omega_f}{\omega_n}$. From Equation 7, the vibration without damping is:

$$x(t) = \frac{X_o}{1-r^2} \sin(\omega_f t); \quad \text{where } X_0 = P_0/k$$

By definition of magnification factor (ratio of X to X_0):

$$MF = \frac{X}{X_o}; \quad \text{where } X = \frac{X_0}{\left|1-r^2\right|} \text{ (per Equation 8)}$$

Then:

$$MF = \frac{1}{|1-r^2|} \quad \text{(MF is always positive so absolute values are used)} \qquad \text{Eq. 10}$$

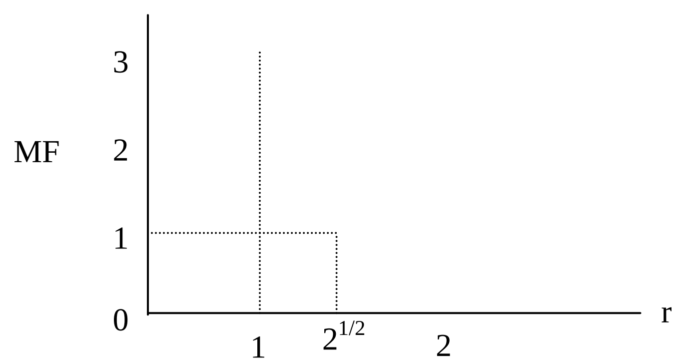
> NOTE: MF is NOT a function of time.
> X = dynamic peak displacement (magnitude of vibration displacement).
> X_o = static displacement created by static force of magnitude P_0.
> Both X and X_0 are magnitudes, they do ***<u>not</u>*** vary with time.

What do you expect the magnification factor to be? Should it always be greater than 1, less than 1, or will it sometime be greater than 1 and sometimes less than 1?

Notice X_0 is not a function of r (ω_f or ω_n) but that X is a function of r. Therefore, MF is a function of the frequency ratio, r. What happens (what is the vibration amplitude) at:

Frequency ratio, $r = \omega_f / \omega_n$	Static displacement, X_0	Actual displacement amplitude, X	Magnification Factor, MF
r = 0 (static load); ($\omega_f = 0$)	X_0		
r = 1 ($\omega_f = \omega_n$)	X_0		
r < 1 ($\omega_f < \omega_n$)	X_0		
$r = 2^{1/2}$	X_0		
r >>1 ($\omega_f >> \omega_n$)	X_0		

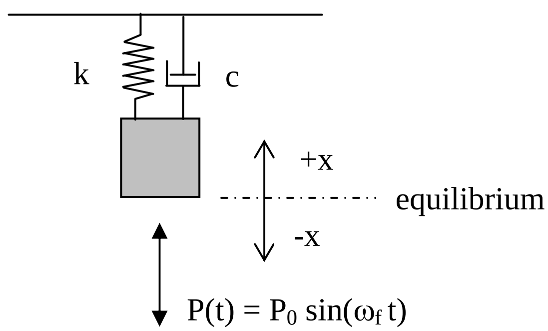
Sketch MF vs. r



ANALYSIS OF FORCED VIBRATION WITH VISCOUS DAMPING (STEADY STATE CONDITION)

The prior section determined the magnification factor for undamped systems. The objective of this section is to determine the magnification factor when viscous damping is present. We will also determine the maximum magnification factor of a given system.

Sketch FBD & kinetic diagram:



Equilibrium equation:

$$m\ddot{x} = -kx - c\dot{x} + P(t)$$

The magnification factor for this condition is:

$$MF = \frac{X}{X_o} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad \text{Eq. 11}$$

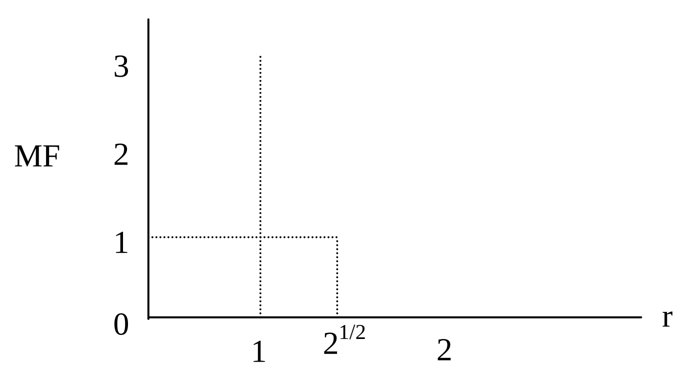
NOTE: **magnification factor is NOT a function of time, it is a constant assuming steady state conditions**. As previously defined:

$$\zeta = \frac{c}{c_c} \qquad r = \frac{\omega_f}{\omega_n}$$

And: $c_c = 2m\omega_n$

Sketch MF for:

a) no damping
b) $\zeta = 0.2$
c) $\zeta = 0.707$



Show that the magnification factor defined in Equation 11, becomes the equation defined in Equation 10 if there is no damping:

Equation 11: $MF = \frac{X}{X_o} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

Equation 10: $MF = \frac{1}{|1-r^2|}$

By using the magnification factor (MF) we can determine the amplitude of vibration for a forced vibration system with or without viscous damping. In real systems, mass, spring constant and damping constant typically do not vary but the forcing frequency (ω_f) will change. For example, your car has a certain mass, springs, and shock absorbers, but the engine runs at different speeds. Therefore, the next question we may want to answer is for a given system (fixed mass, spring, and dashpot), what is the maximum magnification factor (MF_{max}) for any forcing frequency (ω_f)?

For a damping factor greater than 0.707 ($\zeta \geq \frac{1}{\sqrt{2}}$) the magnification factor will remain less than 1 for all values of r. Eq. 12

However, *for* $\zeta < \frac{1}{\sqrt{2}}$ the maximum magnification factor (MF_{max}) is:

$$MF_{max} = \frac{X_{max}}{X_o} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \text{Eq. 13}$$

The last question we will answer is at what frequency ratio (r) will the magnification factor be maximized? The answer depends upon the damping factor:

for $\zeta \geq \frac{1}{\sqrt{2}}$

The magnification factor is maximum at r = 0 (MF_{max} = 1 and occurs at r_{MFmax} = 0). Therefore, with any non-zero value of r, the dynamic response (vibration) will be less than the static displacement due to a static force of P_0 and will decrease with increasing ω_f.

$$for\ \zeta < \frac{1}{\sqrt{2}}$$

The magnification factor will be maximized at the frequency ratio of r_{MFmax}:

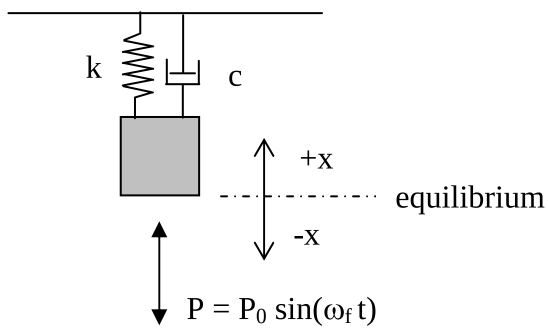
$$r_{MF\,\max} = \sqrt{1-2\zeta^2} \qquad \text{Eq. 14}$$

To summarize, the magnification factor (MF) is the ratio of the displacement amplitude of the vibrating system to the displacement of the system due to a static load of magnitude P_0. It is a function of the system characteristics (mass, spring, dashpot) and the forcing frequency (ω_f). For a given system (with constant mass, spring, dashpot), we can determine the maximum magnification factor for all forcing frequencies (MF_{max}) and we can determine the frequency that maximizes the magnification factor (r_{MFmax}).

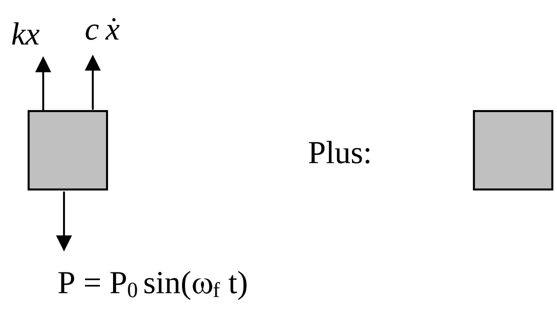
ISOLATION AND TRANSMISSION OF VIBRATION FORCES OF FORCED VIBRATION WITH VISCOUS DAMPING (STEADY STATE CONDITION)

So far we have analyzed the natural frequency of a mass-spring system and determined the vibration amplitude as a function of mass (m), spring (k), dashpot (c), and forcing frequency (ω_f) for a forced vibration system with or without damping. The last question we will answer is what is "what is the maximum force transmitted from the vibrating system to the structure holding it in place?"

We will consider forced vibration with viscous damping. Although there is damping, the system will continue to oscillate indefinitely due to the external force, P. We will investigate the force transmitted by this steady state oscillation.



In addition to the inertial force, there are three forces acting on the mass: the spring force (proportional to displacement), the force from the dashpot (proportional to velocity) and the external force, P. All of these forces are a function of time. The free body diagram (FBD) is:



The equilibrium equation is:

$$m\ddot{x} - kx - c\dot{x} = -P_o \sin(w_f t)$$

The inertial force (m $\ddot{x}$) and the external force (P) are reacted by the spring and dashpot. Only the spring and dashpot are attached to the surrounding structure, and therefore the sum of these two forces is the transmitted force:

$$F_{transmitted} = kx + c\dot{x} \qquad \text{Eq. 15}$$

Although the transmitted force will vary with time (harmonically), we are interested only in the maximum value. **Let F_T be the amplitude of the transmitted force <u>(it does NOT vary with time, it is a constant</u>).** Similarly to the magnification factor (MF), we will normalize the transmitted force amplitude by the static force, P_0. This ratio is the so called transmissibility ratio (TR):

Transmissibility ratio: $TR = \dfrac{F_T}{P_o}$ NOTE: this is NOT a function of time.

We can express the transmissibility ratio as a function of the system's characteristics, m, k, c, and ω_f:

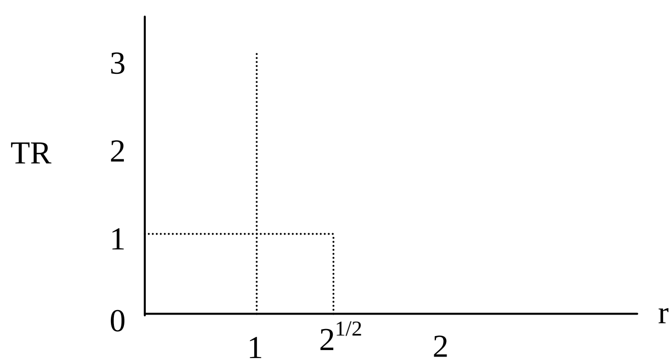
$$TR = \frac{F_T}{P_o} = \frac{\sqrt{k^2 + (c\omega_f)^2}}{\sqrt{(k - m\omega_f{}^2)^2 + (c\omega_f)^2}} \qquad \text{Eq. 16}$$

We can also express it in terms of the normalized characteristics, ζ and r:

$$TR = \frac{F_T}{P_o} = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \qquad \text{Eq. 17}$$

Sketch the transmissibility ratio as a function of frequency ratio for:

a) $\zeta = 0$
b) $\zeta = 0.5$



Note that for all values of damping (ζ), the transmissibility ratio equals one (TR=1) at both r = 0 and at $r = \sqrt{2}$. The maximum transmitted force will always occur at $\omega_f < \omega_n$ regardless of the system characteristics (m, k, c). The frequency ratio for maximizing the transmitted force, is r_{TRmax} and is given by:

TR_{max} occurs at: $$r_{TR\max} = \frac{\sqrt{(1+8\zeta^2)^{1/2} - 1}}{2\zeta}$$ (this is always less than 1). Eq. 18

Example:

Given: An air compressor is placed on an isolation table with spring and dashpots on all 4 corners

The total mass of table and compressor is 800 kg.
Springs: k = 60 KN/m
Dashpot (shock absorbers): c = 4000 kg/sec

Forcing function is given as $P(t) = P_o sin(\omega_f t)$
where ω_f = 370 rad/sec (3600 rpm) and P_o = 500N

Find: a) Damping ratio, ζ
b) What forcing frequency (ω_f) results in maximizing the displacement?
c) Determine the vibration displacement amplitude at the forcing frequency in part b.
d) What forcing frequency (ω_f) results in maximizing the transmitted force?
e) Determine the transmitted force at the forcing frequency in part d.

Assume:

Sketch:

Solution:

CONCLUSION & SUMMARY

We have studied the following systems for one degree of freedom:

- **Analysis of simple harmonic motion of undamped free vibration**
  - The natural frequency if a function of mass and spring constant:
  - $\omega_n = \sqrt{k/m}$

- **Basic behavior of of viscous damped free vibration system.**
  - When a damped system is displaced from equilibrium, it will seek out equilibrium and eventual will come to rest at equilibrium (unless acted upon by a varying external force).
  - If it is under-damped, it will oscillate before coming to rest.
  - If it is critically damped, it will not oscillate (if initial velocity is zero) but rather it will approach equilibrium asymptotically.
  - If it is over-damped, it will not oscillate (if initial velocity is zero) and will take a longer time to reach equilibrium than a critically damped system.

- **Analysis of forced vibration without dampening.**
  - The vibration amplitude (*X*) of forced vibration without dampening is a function of the system characteristics (m and k) as well as the forcing frequency (ω_f). The ratio of the amplitude normalized by the displacement created by a static force, P_0 is called the magnification factor, MF. Although the system oscillates (*x(t)*), the magnification factor is based on the amplitudes and, therefore, is constant for the system as long as the forcing frequency (ω_f) remains constant.
  - $MF = \dfrac{X}{X_0} = \dfrac{1}{|1-r^2|}$

- **Analysis of forced vibration with viscous damping.**
  - The magnification factor for a system with viscous damping is defined in the same way as for no damping, but the equation is more complex.
  - $MF = \dfrac{X}{X_o} = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$
  - Forζ < 0.707, the magnification factor will remain less than 1 for all values of r.

  - For ζ < 0.707, the maximum magnification factor for a given system (m, k, c) for all frequency ratios (r) is:

    $$MF_{\max} = \frac{X_{\max}}{X_o} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

    And occurs at: $r_{MF\max} = \sqrt{1-2\zeta^2}$

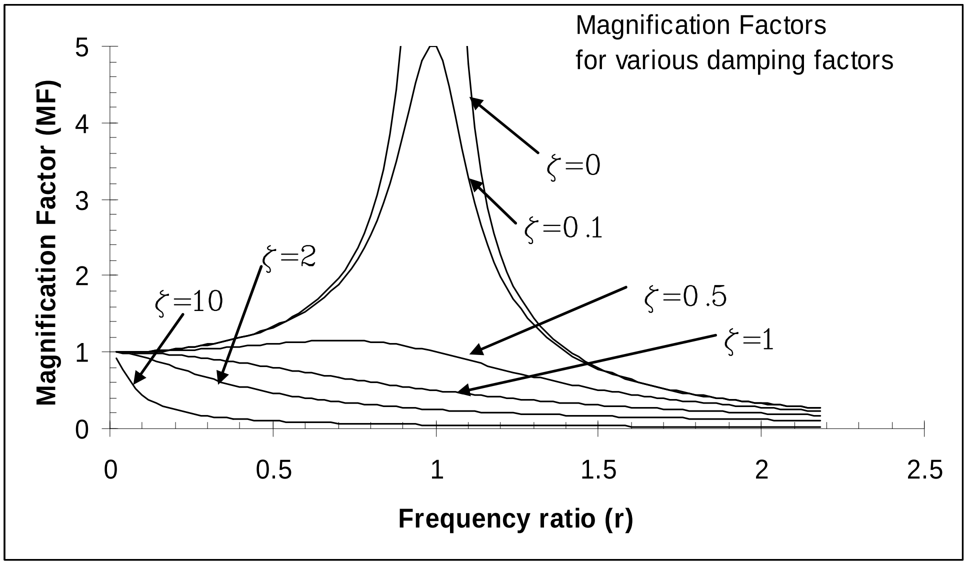
- **Isolation and transmission of vibration forces of forced vibration with viscous damping.**
  - The amplitude of the force transmitted from the vibrating system to the surrounding support structure is F_T. By normalizing F_T by the static force, P_0, the transmissibility ratio, TR, is described as:

$$TR = \frac{F_T}{P_o} = \frac{\sqrt{k^2 + (c\omega_f)^2}}{\sqrt{(k - m\omega_f^{\,2})^2 + (c\omega_f)^2}} = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

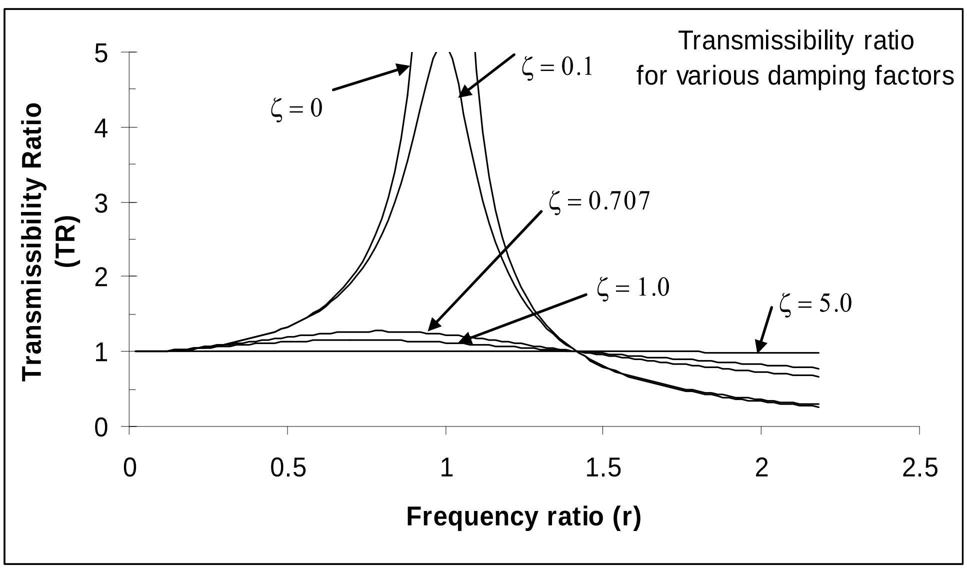
  - Transmissibility ratio does NOT vary with time; it is a constant for a given system and forcing frequency. Even though there is viscous damping, the system is assumed to have steady state oscillation due to harmonic external force, P.
  - For a given system (m, k, c) the transmissibility ratio is a function of the forcing frequency, and it becomes maximum at the frequency ratio of r_{TRmax}:

$$r_{TR\max} = \frac{\sqrt{(1 + 8\zeta^2)^{1/2} - 1}}{2\zeta}$$

  - r_{TRmax} is always less than 1 regardless of system characteristics (m, k, c). In other words, the maximum transmitted forces for a system will always occur when the forcing frequency is less than the natural frequency.



$$MF = \frac{X}{X_o} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$



$$TR = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2 + 2\zeta r)^2}}$$