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Associated primes over Ore extensions

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Abstract

For a module M_R we compute the set of associated primes of $M[x;\sigma]$ over the left Ore extension $R[x;\sigma]$ for any surjective endomorphism σ of R. This result leads to necessary and sufficient conditions under which the associated primes of $M[x;\sigma]$ are precisely the extensions of the associated primes of M. We relate these results to previous work regarding the propagation of prime ideals of $R[x;\sigma]$ and include several illustrative examples.

Keywords: Associated primes; Ore extension; Skew polynomial

1. Introduction

Let R be a ring with identity and let σ be an endomorphism of R. Consider $S = R[x;\sigma]$, the left Ore extension. We use the convention that coefficients are written on the left and the defining relation is $xr = \sigma(r)x$ [1,3]. One question which arises in this construction is how the prime ideals of $R[x;\sigma]$ are built from ideals of R. Much of the initial work regarding the propagation of primes when R is commutative was done by Irving in [4]. The technique pioneered by Irving and modeled by others was to first define the notion of a 'σ-prime' ideal. This lead to several different inequivalent definitions, all of which are based on the usual noncommutative definitions for prime ideals. Conditions were then given under which, given a σ-prime ideal, $I \leqslant R$, one can conclude IS is prime. We have

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chosen a slightly different tack. This paper grew out of an attempt to place a more noncommutative framework on previous work [1], which related the associated prime ideals of a right R-module, M, to the associated primes of $M[x;\sigma]_S$. We quickly realized this relationship was greatly simplified by assuming that σ is surjective. Indeed, with that hypothesis, results relating the set of annihilator ideals of M_R and the set of associated primes of $M[x;\sigma]_S$ can then be read off easily. Using these results, one is much better off simply computing the associated primes of $M[x;\sigma]$ directly. As a corollary to our main result, we show that an ideal I for which $I[x;\sigma]$ is prime is precisely an ideal we define as the σ-associated ideal to some σ-prime module N_R.

In this section we provide the definitions and statements of the main result and its corollaries. In the second section, we discuss several examples outlining the use of these results. The last section is devoted to the proofs of several preliminary results and then the proofs of the principal results. Before continuing with the development of the main results, I would like to thank the referee for suggestions which have substantially improved this paper.

With minimal notation we can state our main result. We recall, in general, that a nonzero submodule $N < M$ is *prime* if $\mathrm{ann}_R(N')$ is constant across all nonzero submodules of N and in such cases, $\mathrm{ann}(N)$ is necessarily a prime ideal. Also a left, right or two-sided ideal I is a *σ-ideal* if $\sigma(I) \subseteq I$ [3], and is called a *σ-invariant* ideal if $I = \sigma^{-1}(I)$ [4].

Definition 1.1. For any subset $I \subseteq R$, let $I_\sigma = \bigcap_{j\in\mathbb{N}} \sigma^{-j}(I)$. We say that a nonzero submodule $N < M$ is a *σ-prime* submodule if $(\mathrm{ann}(N'))_\sigma$ is constant over nonzero submodules of N and additionally $\sigma^{-1}((\mathrm{ann}(N))_\sigma) \subseteq (\mathrm{ann}(N))_\sigma$.

We note that neither $\mathrm{ann}(N)$ nor $(\mathrm{ann}(N))_\sigma$ need be prime for a σ-prime submodule N. Nevertheless, when N is σ-prime we refer to $I = (\mathrm{ann}(N))_\sigma$ as a *σ-associated* ideal of M and let σ-$\mathrm{Ass}(M)$ denote the set of σ-associated ideals. If I is a σ-associated ideal, by definition $\sigma^{-1}(I) \subseteq I$. Moreover, $I = (\mathrm{ann}(N))_\sigma$ for some σ-prime submodule N, so $I = \bigcap_{j\in\mathbb{N}} \sigma^{-j}(\mathrm{ann}(N)) \subseteq \bigcap_{j>0} \sigma^{-j}(\mathrm{ann}(N)) = \sigma^{-1}((\mathrm{ann}(N))_\sigma) = \sigma^{-1}(I)$. Thus a σ-associated ideal is σ-invariant.

If I is a subset of R we write $I[x;\sigma]$ for the set of polynomials in $R[x;\sigma]$ whose (left) coefficients are all in I. Even if I is an ideal of R, $I[x;\sigma]$ need not be an ideal in $R[x;\sigma]$.

Theorem 1.2. *Let R be a ring with identity and let σ be a surjective endomorphism. For any right R-module M,* $\mathrm{Ass}(M[x;\sigma]) = \{I[x;\sigma] \mid I \in \sigma\text{-}\mathrm{Ass}(M)\}$.

It is apparent that $I[x;\sigma]$ can be an associated prime of $M[x;\sigma]$ when I is not a prime of R. More remarkably, I need not be the annihilator of a submodule of M, as illustrated in Example 2.1. However, the following corollary shows that σ-associated ideals are precisely those ideals which extend to prime ideals.

Corollary 1.3. *Suppose σ is surjective. Then the following are equivalent conditions on an ideal $I \leqslant R$.*

(1) *I is the σ-associated ideal to some σ-prime module N.*
(2) *$I[x;\sigma]$ is a prime ideal of S.*

Recall that a module is *σ-compatible* if all annihilators of elements of M are σ-invariant σ-ideals. This was shown to be a sufficient condition to conclude that the associated primes of $M[x;\sigma]$ are precisely the extensions the associated primes of M [1]. However, we observe that any σ-invariant associated prime is automatically a σ-associated ideal. Consequently, the associated primes of $M[x;\sigma]$ coincide with the extensions of the associated primes of M precisely when every associated prime of M is σ-invariant and every other annihilator ideal I is either not σ-invariant, or satisfies the condition that for every submodule N with $\operatorname{ann}(N) = I$, there exists $0 \neq K < N$ such that $(\operatorname{ann}(K))_\sigma \neq I$. In light of this, the analogue of the main result of [1], is clear:

Corollary 1.4. *Suppose σ is surjective and M is a right R-module.*

(1) *If $\mathfrak{p} \in \operatorname{Ass}(M)$, then $\mathfrak{p}[x;\sigma] \in \operatorname{Ass}(M[x;\sigma]_S)$ if and only if $\mathfrak{p}$ is σ-invariant.*
(2) *If M is σ-compatible, or more generally, if every annihilator of a submodule of M is σ-invariant, then $\operatorname{Ass}(M[x;\sigma]_S) = \{\mathfrak{p}[x;\sigma] \mid \mathfrak{p} \in \operatorname{Ass}(M)\}$.*

When R is Noetherian, σ is an automorphism, and we get a much stronger result:

Corollary 1.5. *If R is a Noetherian ring, then $\operatorname{Ass}(M[x;\sigma]_S) = \{\mathfrak{p}_\sigma[x;\sigma] \mid \mathfrak{p} \in \operatorname{Ass}(M)\}$.*

Remark 1.6. Results parallel to Theorem 1.2 and its corollaries for the skew-Laurent polynomial extensions are made by altering Definition 1.1. In order to define $R[x,x^{-1};\sigma]$, σ must be an automorphism. When σ is an automorphism and $I \subseteq R$ define $I_{\sigma^*} = \bigcap_{j\in\mathbb{Z}} \sigma^j(I)$. We call a nonzero submodule $N \leqslant M$ *Laurent σ-prime* if $(\operatorname{ann}(N'))_{\sigma^*}$ is constant over all nonzero submodules of N and call an ideal I a *Laurent σ-associated ideal* of M if $I = (\operatorname{ann}(N))_{\sigma^*}$ for some Laurent σ-prime submodule N. Although this altered definition only applies for the skew-Laurent extensions, the Laurent-polynomial version of Theorem 1.2 is now easily deduced: $\operatorname{Ass}(M[x,x^{-1};\sigma]_{R[x,x^{-1};\sigma]}) = \{I[x,x^{-1};\sigma] \mid I$ is a Laurent σ-associated ideal of $M\}$. The corollaries of this are also straightforward. Observe that for any left, right, or two-sided ideal I, I_{σ^*} is σ-invariant. Thus for every $\mathfrak{p} \in \operatorname{Ass}(M)$, $\mathfrak{p}_{\sigma^*}$ is automatically a Laurent σ-associated ideal of M. Therefore $\{\mathfrak{p}_{\sigma^*}[x,x^{-1};\sigma] \mid \mathfrak{p} \in \operatorname{Ass}(M)\} \subseteq \operatorname{Ass}(M[x,x^{-1};\sigma]_{R[x,x^{-1};\sigma]})$. Moreover, the notion of a σ-prime ideal in the Laurent extension case is well established. A σ-prime ideal is a σ-invariant ideal, P, which satisfies the condition that if I, J are σ-invariant ideals with $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$. Such ideals always extend to prime ideals of $R[x,x^{-1};\sigma]$ [5]. In particular, the analogue of Corollary 1.3 is that an ideal is σ-prime if and only if it is the Laurent σ-associated ideal of some Laurent σ-prime module. The proofs for the results for the Laurent extensions are similar to the proofs of the main result that appear in Section 3 and are therefore omitted.

2. Examples

A module can easily fail to be σ-compatible, as defined in [1], but still have each associated prime of $M[x;\sigma]$ be extended from one of M. For example, if R is any simple ring with automorphism σ, then 1.2 shows that for any nontrivial module M, $\{(0)\} = \mathrm{Ass}(M[x;\sigma]_S) = \{\mathfrak{p}_\sigma[x;\sigma] \mid \mathfrak{p} \in \mathrm{Ass}(M)\}$. However, if R has a proper nonzero right ideal J which is not σ-invariant, then $M = R/J$ is not σ-compatible.

It is more interesting, in light of Theorem 1.2, to investigate modules, M, for which the associated primes of M fail to extend. The first of these investigations involves an associated prime which is not σ-invariant. Throughout the next examples we let k denote a field.

Example 2.1. Let $R = k[s,t]$ and $M_R = R/(t)$. Let σ be the k-algebra automorphism of R transposing s and t. Clearly $\mathrm{Ass}(M_R) = \{(t)\}$. But (t) is not σ-invariant since $\sigma^{-1}((t)) = (s)$. Now $(t)_\sigma = (t) \cap (s) = (st)$. We observe that $(st) \in \sigma\text{-}\mathrm{Ass}(M)$ and so by 1.2, $\mathrm{Ass}(M[x;\sigma]) = \{(st)[x;\sigma]\}$. Note that (st) is not prime, and more, is not the annihilator of any submodule of M.

For $\mathfrak{p}[x;\sigma] \in \mathrm{Ass}(M[x;\sigma]_S)$, $\mathfrak{p}$ need not be a prime ideal of R. However, in the above example, $(t)_\sigma = (st)$, so one question that arises is whether or not $\mathfrak{p}_\sigma$ is a σ-associated ideal when $\mathfrak{p}$ is prime. The following example shows that it need not be, even when R is commutative. Note that by 1.5 we must begin with a non-Noetherian base ring.

Example 2.2. Let $R = k[\ldots, t_{-1}, t_0, t_1, \ldots]$, and $M_R = R/(\ldots, t_{-1}, t_0)$. Consider the k-algebra automorphism of R given by $\sigma(t_i) = t_{i-1}$ for all i. Clearly M is prime with annihilator $(\ldots, t_{-1}, t_0)$. Thus $\mathrm{Ass}(M) = \{(\ldots, t_{-1}, t_0)\}$. Observe that

$$\begin{aligned}(\ldots, t_{-1}, t_0)_\sigma &= (\ldots, t_{-1}, t_0) \cap (\ldots, t_{-1}, t_0, t_1) \cap (\ldots, t_{-1}, t_0, t_1, t_2) \cap \cdots \\ &= (\ldots, t_{-1}, t_0),\end{aligned}$$

but $(\ldots, t_{-1}, t_0)$ is not σ-invariant, hence not a σ-associated ideal. Therefore by Theorem 1.2, $\mathrm{Ass}(M[x;\sigma]_S) = \emptyset$. Thus an associated prime of M_R need not extend in any meaningful way to an associated prime of $M[x;\sigma]_S$.

The next example illustrates that a nonprime annihilator I can be a σ-associated ideal which is not $\mathfrak{p}_\sigma$ for any associated prime $\mathfrak{p}$.

Example 2.3. Let $R = k[\ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots]/(t_i^2)$, and define $\bar{t_i} = t_i + (t_i^2)$. Set $M_R = R_R$ and let σ be the k-algebra automorphism of R given by $\sigma(\bar{t_i}) = \bar{t}_{i-1}$ for all i. We claim that M is σ-prime, but note that it is not prime. Observe that $\mathrm{ann}(M) = 0$ is σ-invariant. Thus $(\mathrm{ann}(M))_\sigma = 0$. In order to show M is σ-prime it will be enough to show, for any $g \in (\bar{t_i})_{i\in\mathbb{Z}}$, that $(g)_\sigma = 0$. If $g \in (\bar{t_i})_{i\in\mathbb{Z}}$, then there exist $j_1, j_2, \ldots, j_n \in \mathbb{Z}$ such that $g \in (\bar{t}_{j_1}, \bar{t}_{j_2}, \ldots, \bar{t}_{j_n})$. Now $(g)_\sigma \subseteq (\bar{t}_{j_1}, \bar{t}_{j_2}, \ldots, \bar{t}_{j_n})_\sigma = 0$. Therefore M is σ-prime. Thus 0 is the only σ-prime ideal of M. Therefore by Theorem 1.2, $\mathrm{Ass}(M[x;\sigma]) = \{0\}$. In contrast, we observe that every nonzero cyclic submodule $fR \leqslant M$ contains a nonzero cyclic submodule whose annihilator strictly contains $\mathrm{ann}(fR)$. That is, M has no cyclic prime submodules, hence no prime submodules. Therefore $\mathrm{Ass}(M) = \emptyset$.

3. Proofs of the main results

The proof of Theorem 1.2 relies on some elementary initial results. The first result is well known in commutative algebra. We generalize the result found in [2]; the proof here is quite different.

Proposition 3.1. *If $\mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_i$ is $\mathbb{Z}$-graded ring with identity, $\mathcal{M}_A = \bigoplus_{i\in\mathbb{Z}} \mathcal{M}_i$ is a graded module, $\mathcal{N} \leqslant \mathcal{M}$ is a prime submodule and $\mathfrak{q} = \operatorname{ann}(\mathcal{N})$, then $\mathfrak{q}$ is a homogeneous ideal.*

Proof. Let $a = a_0 + \cdots + a_k \in \operatorname{ann}(\mathcal{N})$, where each a_i is a nonzero element of $\mathcal{A}_{m_i}$ for some integers $m_0 < \cdots < m_k$. It will be enough to show that $a_0 \in \operatorname{ann}(\mathcal{N})$. It will then follow by induction on k that $\mathcal{N}a_i = 0$ for each i, and so the homogeneous terms of a belong to $\operatorname{ann}(\mathcal{N})$.

Let $m \in \mathcal{N}$ be an element of least possible length. That is, every element is the unique sum of nonzero homogeneous elements, and for m, it involves the least number of terms possible among elements of $\mathcal{N}$. Write $m = m_0 + \cdots + m_l$, where each m_i is a nonzero element of M_{n_i} for some integers $n_0 < \cdots < n_l$. Clearly, for any homogeneous component, $\mathcal{A}_r$, every nonzero element of $m\mathcal{A}_r$ has length l. However, a_0 annihilates the first term of every nonzero element of $m\mathcal{A}_r$, hence every nonzero element of $m\mathcal{A}_r a_0$ has length less than l. By the minimality of l, it must be that $m\mathcal{A}_r a_0 = 0$. Thus $m\mathcal{A}a_0 = 0$. As $\mathcal{N}$ is prime, $a_0 \in \operatorname{ann}(m\mathcal{A}) = \operatorname{ann}(\mathcal{N})$. $\square$

Corollary 3.2. *If $\mathfrak{q} \in \operatorname{Ass}(M[x;\sigma]_S)$, then $\mathfrak{q} = I[x;\sigma]$ for some σ-invariant ideal $I \leqslant R$.*

Proof. We grade $S = R[x;\sigma]$ and $M[x;\sigma]_S$ by degree in x. The preceding proposition shows $\mathfrak{q}$ is homogeneous with respect to this grading. Since $M[x;\sigma]$ is x-torsionfree, it follows that $\mathfrak{q} = I[x;\sigma]$ for some ideal I. To show I is σ-invariant, let $\mathcal{N} \leqslant M[x;\sigma]$ be prime with annihilator $I[x;\sigma]$. On one hand, $0 \neq \mathcal{N}x \leqslant \mathcal{N}$, so $0 = \mathcal{N}xI = \mathcal{N}\sigma(I)x$. Thus $\sigma(I) \subseteq I$, which says $I \subseteq \sigma^{-1}(I)$. On the other, $0 = \mathcal{N}Ix \supseteq \mathcal{N}\sigma(\sigma^{-1}(I))x = Nx(\sigma^{-1}(I))$. Since $\mathcal{N}$ is prime, $\operatorname{ann}(\mathcal{N}x) = I[x;\sigma]$. Consequently, $\sigma^{-1}(I) \subseteq I$. Therefore $\sigma^{-1}(I) = I$. $\square$

Corollary 3.3. *If σ is surjective and $N \leqslant M[x;\sigma]_S$ is prime, then $\operatorname{ann}_S(N) = I[x;\sigma]$ where I is the σ-associated ideal of a σ-prime submodule of M.*

Proof. By the previous corollary, $\operatorname{ann}_S(N) = I[x;\sigma]$ where $I \leqslant R$ is σ-invariant. Let $0 \neq f \in N$ be of minimal length l, and write $f = m_0x^{a_0} + \cdots + m_lx^{a_l}$, where each m_i is a nonzero element of N and $a_0 < \cdots < a_l$. We show m_0R is σ-prime with σ-associated ideal I.

Set $a = a_0$ and let $m \in m_0R$. Since σ is onto, we may select $r \in R$ so that $m_0\sigma^a(r) = m$. Let $J = \operatorname{ann}_R(mR)$. Since $fS \subseteq N$ and $I \subseteq \operatorname{ann}(N)$, $fSI = 0$, and so $mR\sigma^a(I) = 0$. As I is σ-invariant and σ is onto, $mRI = 0$. Thus $I \subseteq J$.

Observe that, for all $i \geqslant 0$, every nonzero element of $frRx^i$ has length l. Every element of $frRx^i(J_\sigma)$ has length less than l, so $frRx^iJ_\sigma = 0$, by the minimality of l. Since

N is prime $J_\sigma S \subseteq \operatorname{ann}_S(frS) = I[x;\sigma]$. Therefore $J_\sigma = I$ and we conclude m_0R is σ-prime. □

Lemma 3.4. *For an R-module N, $\operatorname{ann}_S(N[x;\sigma]) = (\operatorname{ann}_R(N))_\sigma[x;\sigma]$.*

Proof. Since $N[x;\sigma]$ is homogeneous, $\operatorname{ann}_S(N[x;\sigma])$ is homogeneous. Let $rx^i \in \operatorname{ann}_S(N[x;\sigma])$. Then $Nx^jr = 0$ for all $j \geqslant 0$, or, equivalently, $r \in \sigma^{-j}(\operatorname{ann}_R(N))$ for all $j \geqslant 0$. That is, $rx^i \in \operatorname{ann}_S(N[x;\sigma])$ if and only if $r \in (\operatorname{ann}_R(N))_\sigma$. □

Lemma 3.5. *Let σ be surjective. Then N_R is σ-prime with σ-associated ideal I if and only if $N[x;\sigma]$ is prime with associated prime $I[x;\sigma]$.*

Proof. Suppose $N[x;\sigma]$ is prime with associated prime $I[x;\sigma]$. By Lemma 3.4, $I = (\operatorname{ann}(N))_\sigma$. Let $m \in N$ and set $J = \operatorname{ann}_R(mR)$. Since $N[x;\sigma]$ is prime, $I[x;\sigma] = \operatorname{ann}_S((mR)[x;\sigma]) = J_\sigma[x;\sigma]$. Thus $J_\sigma = I$ and so N is σ-prime with σ-associated ideal I.

Conversely, if N is σ-prime with σ-associated ideal I, then 3.4 shows $I[x;\sigma] = \operatorname{ann}_S(N[x;\sigma])$. If $N[x;\sigma]$ is not prime, then there exists a nonzero element $f \in N[x;\sigma]$ such that $J = \operatorname{ann}_S(fS)$ strictly contains $I[x;\sigma]$. Write $f = m_0x^{a_0} + \cdots + m_kx^{a_k}$, where $m_i \neq 0$ and $a_0 < \cdots < a_k$ and let J_0 be the set of constant coefficients from elements of J. Select a nonzero element $s \in J \setminus I[x;\sigma]$ of minimal length. Observe that if $s = r_0x^{b_0} + r_1x^{b_1} + \cdots + r_mx^{b_m} \in J$, then $r_0 + r_1x^{b_1-b_0} + \cdots + r_mx^{b_m-b_0} \in J$, as x acts without torsion. Thus $r_0 \in J_0$. Since s is of minimal length, $r_0 \notin I$ and so J_0 strictly contains I.

However, every element of J_0 annihilates the term of lowest degree of every element of fS. In particular $m_0x^{a_0}Rx^jJ_0 = 0$ for all $j \geqslant 0$. Thus $J_0 \subseteq \bigcap_{j \geqslant a} \sigma^{-j}(\operatorname{ann}(m_0R)) = \sigma^{-a_0}((\operatorname{ann}(m_0R))_\sigma) = \sigma^{-a_0}(I)$. Since I is σ-invariant, this implies $J_0 \subseteq I$, a contradiction. Therefore no such f exists, and $N[x;\sigma]$ is prime with associated prime $I[x;\sigma]$. □

We now have all of the preliminary results needed for the proof of the main result. The proof hinges on the fact that we already know what form the associated primes must take.

Proof of Theorem 1.2. If $\mathfrak{p} \in \operatorname{Ass}(M[x;\sigma])$, then $\mathfrak{p} = \operatorname{ann}(N)$ for some prime submodule $N \leqslant M[x;\sigma]$. By Corollary 3.3, $\mathfrak{p} = I[x;\sigma]$, where I is the σ-associated ideal of a σ-prime submodule of M.

Conversely, if I is a σ-associated ideal of M, then $I = (\operatorname{ann}(L))_\sigma$ for some σ-prime submodule $L \leqslant M$. By Lemma 3.5, $I[x;\sigma] \in \operatorname{Ass}(M[x;\sigma])$ as it is the annihilator of the prime submodule $L[x;\sigma] \leqslant M[x;\sigma]$. □

Proof of Corollary 1.3. If I is the σ-associated ideal to a σ-prime module, N_R, then 3.5 shows $I[x;\sigma]$ is prime. Conversely, suppose $I[x;\sigma] \leqslant S$ is prime. Then $\sigma(I) \subseteq I$ since $xI[x;\sigma] \subseteq I[x;\sigma]$. So $I \subseteq \sigma^{-1}(I)$. As σ is surjective, $(xS)(\sigma^{-1}(I)S) = SIxS \subseteq I[x;\sigma]$. Since $I[x;\sigma]$ is prime, $\sigma^{-1}(I)S \subseteq I[x;\sigma]$. Thus $\sigma^{-1}(I) \subseteq I$. Therefore I is σ-invariant. According to 3.5, it will be enough, to show $N = (R/I)[x;\sigma]$ is prime with associated prime $I[x;\sigma]$. Note that since I is σ-invariant, $N \cong S/(I[x;\sigma])$. Clearly,

$I[x;\sigma] = \mathrm{ann}_S(N)$. Let $f \in R[x;\sigma] \setminus I[x;\sigma]$ and set $J = \mathrm{ann}_S((f + I[x;\sigma])S)$. Then $(f + I[x;\sigma])SJ \subseteq fSJ + I[x;\sigma]J \subseteq I[x;\sigma]$. So $fSJ \subseteq I[x;\sigma]$. Since $I[x;\sigma]$ is prime, $J \subseteq I[x;\sigma]$. Thus $J = I[x;\sigma]$ as required. □

Proof of Corollary 1.4. To verify (1), suppose $\mathfrak{p} \in \mathrm{Ass}(M)$ and let $N \leqslant M$ be a prime submodule with $\mathrm{ann}(N) = \mathfrak{p}$. N is automatically σ-prime with $\mathfrak{p}_\sigma$ as its σ-associated ideal, whenever $\mathfrak{p}_\sigma$ is σ-invariant. Consequently, if $\mathfrak{p}$ is σ-invariant, Lemma 3.5 shows $\mathfrak{p}[x;\sigma] \in \mathrm{Ass}(M[x;\sigma])$. Conversely, if $\mathfrak{p}[x;\sigma] \in \mathrm{Ass}(M[x;\sigma])$, then 3.2 shows that $\mathfrak{p}$ is a σ-associated ideal and is therefore σ-invariant.

For (2), we observe that the hypotheses along with (1) imply $\{\mathfrak{p}[x;\sigma] \mid \mathfrak{p} \in \mathrm{Ass}(M)\} \subseteq \mathrm{Ass}(M[x;\sigma])$. Suppose I is an ideal, $I \notin \mathrm{Ass}(M)$, but is the annihilator of a nonzero submodule $N \leqslant M$. Since $I \notin \mathrm{Ass}(M)$, we may assume no such submodule is prime, and so contains a nonzero submodule L whose annihilator J strictly contains I. Since $I_\sigma = I$ and $J_\sigma = J$ by hypothesis, I cannot be a σ-associated ideal. This proves $\{\mathfrak{p}[x;\sigma] \mid \mathfrak{p} \in \mathrm{Ass}(M)\} = \mathrm{Ass}(M[x;\sigma])$. □

Proof of Corollary 1.5. Suppose R is Noetherian and let $\mathfrak{q} \in \mathrm{Ass}(M[x;\sigma]_S)$. Then $\mathfrak{q} = I[x;\sigma]$ for some σ-prime ideal of M. In particular, there exists a σ-prime submodule $N \leqslant M$ with $I = (\mathrm{ann}(N))_\sigma$. Since R is Noetherian, there exists an ideal $\mathfrak{p}$ which is maximal among annihilators of nonzero submodules of N. We know $\mathfrak{p}$ is an associated prime of N, and hence of M. Moreover, since N is σ-prime, $\mathfrak{p}_\sigma = I$.

Conversely, suppose $\mathfrak{p} \in \mathrm{Ass}(M)$ and let $L \leqslant M$ be a prime submodule with annihilator $\mathfrak{p}$. Set $I = \mathfrak{p}_\sigma$, and note that $\sigma(I) \subseteq I$. Since σ is an automorphism and R is Noetherian, this implies that I is σ-invariant. Therefore L is σ-prime with σ-associated ideal I. By 3.4, $L[x;\sigma]$ is a prime submodule of $M[x;\sigma]$ with associated prime ideal, $I[x;\sigma]$. □

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