Mth 201
Related Rates Practice Problems

1. A spherical snowball is melting. Its radius is decreasing at 0.2 centimeters per hour when the radius is 15cm. How fast is the volume decreasing at this time?

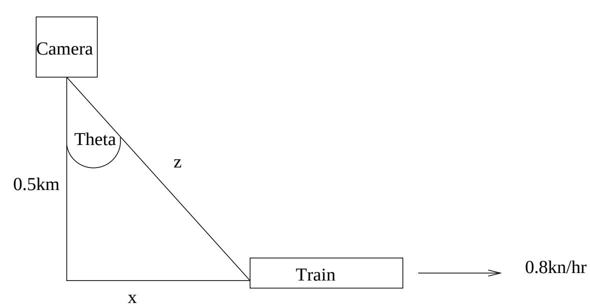
   Solution:

   - Variables Volume: V, radius: r and time: t
   - We want dV/dt - rate of change of volume with respect to time.
   - We know $dr/dt = -0.2$ - the rate of change of radius with respect to time (it is negative since the radius is decreasing).
   - Equations relating variables: $V = 4\pi r^3/3$ (volume of a sphere in terms of radius).
   - Solving the problem: We want dV/dt, so we need to differentiate both sides with respect to t. Differentiating, we get

   $$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

   Plugging in $r = 15$ and $dr/dt = -0.2$, we get $dv/dt = 4\pi(15)^2(-0.2) \sim -565.5cm^3/hr$.

2. A train is traveling at 0.8km/min along a straight track, moving in the direction as shown below. A movie camera, 0.5km away from the track is focused on the train.



   (a) Express z, the distance between the camera and the train as a function of x.

   Solution: Using Pythagoras Theorem, we have $x^2 + (0.5)^2 = z^2$, so

   $$z = \sqrt{x^2 + \frac{1}{4}}.$$

(b) How fast is the distance from the camera to the train changing when the train is 1km from the camera? Give units.

Solution:

- Variables: The distance from the camera to the train: z, the distance x (which is the distance from where the train directly passes the camera), time: t
- We want dz/dt - the rate of change of distance from the camera to the train.
- We know $dx/dt = 0.8$ since the train is moving at $0.8km/hr$ (so the distance x is increasing at a rate of $0.8km/hr$.
- Equations relating variables:

$$x^2 + \frac{1}{4} = z^2.$$

- Solving the problem: Since we want dz/dt, we differentiate the expression we have found with respect to t:

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

giving

$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt}.$$

When $z = 1$, we have $x = \sqrt{3}/2$ so $dz/dt = 0.8\sqrt{3}/2$.

(c) How fast is the camera rotating (in radians/min) at the moment when the train is 1km from the camera?

Solution:

- Variables: In this case, instead of the variable z, we introduce the variable θ - all other variables are the same as those for the first question.
- We want $d\theta/dt$
- We know $dx/dx = 0.8$
- Equations relating variables: Using simple trigonometry, we have $\tan(\theta) = 2x$.
- Solving the problem: Differentiating with respect to time t, we have

$$\frac{1}{\cos^2(\theta)}\frac{d\theta}{dt} = 2\frac{dx}{dt}$$

or

$$\frac{d\theta}{dt} = 2\cos^2(\theta)\frac{dx}{dt}.$$

When $z = 1$ and $x = 0.5$, then $\theta = \pi/3$ (using simple trigonometry), so $d\theta/dt = 2 \cdot (0.5)^2 \cdot 0.8 = 0.4rads/min$.

3. A water tank is in the shape of an inverted cone with depth 10m and top radius 8m. Water is flowing into the tank at $0.1\text{m}^3/\text{min}$ but leaking out at a rate of of $0.001h^2\text{m}^3/\text{min}$ where h is the depth of the water in the tank. Can the tank ever overflow? Explain.

   Solution:

   This looks like a related rates problem, but can in fact be solved without using related rates. Observe that dV/dt measures the rate of change of volume. Since the rate of change of volume will be equal to the difference between what is flowing into the tank and what is flowing out of the tank, we have

   $$\frac{dV}{dt} = 0.1 - 0.001h^2$$

   where h is the height of the water in the tank. Notice that when $h = 10$ (so it is at the top of the tanks), we have $dV/dt = 0$, meaning there is no change in the height of the water. This means that when the water reaches the top of the tank, the amount of water will neither increase or decrease, but stay steady, so it follows that the tank will never overflow.

4. A ruptured oil tanker causes a circular oil slick on the surface of the ocean. When its radius is 150m, the radius of the slick is expanding by 0.1 meters/minute and its thickness is 0.02 meters. At that moment:

   (a) How fast is the area of the slick expanding?

   Solution:

   - Variables: We need radius r of the slick, area A and time t.
   - We want dA/dt
   - We know $dr/dt = 0.1$
   - Equations relating variables: We can use the area formula for a circle - $A = \pi r^2$.
   - Solving the problem: Since we are trying to find dA/dt, we differentiate the area formula with respect to t. Specifically, we get
     $$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}.$$
     Evaluating when $r = 150$ and $dr/dt = 0.1$, we get $dA/dt = 2\pi \cdot 150 \cdot 0.1 = 94.25$.

   (b) The circular slick has the same thickness everywhere, and the volume of oil spilled remains fixed. How fast is the thickness of the slick decreasing?

   Solution:

- Variables: We shall need radius r (since we are told dr/dt), thickness T and time t.
- We want dT/dt
- We know dr/dt
- Equations relating variables: Since the volume is always fixed, we have $V = \pi \cdot (150)^2 \cdot 0.2 = 14137$. In general, the volume will be equal to $\pi \cdot r^2 \cdot T$, so we have $\pi \cdot r^2 \cdot T = 14137$.
- Solving the problem: Differentiating the equation for volume (using the product rule), we have

  $$2\pi \cdot r \cdot T\frac{dr}{dt} + \pi r^2 \frac{dT}{dt} = 0$$

  or

  $$\frac{dT}{dt} = -\frac{2T}{r}\frac{dr}{dt}.$$

  Evaluating when $T = 0.2$, $r = 150$ and $dr/dt = 0.1$, we have $dT/dt = -2 * 0.2/150 * 0.1 = -0.000267 cm/hr$.

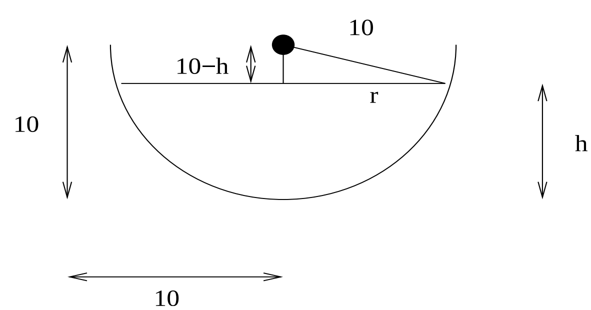
5. (a) A hemispherical bowl of radius 10cm contains water to a depth of $h$cm. Find the radius of the surface of the water as a function of h.

The following sketch shows how we can relate r and h - specifically, we shall use the triangle illustrated. We have

$$r^2 + (10 - h)^2 = r^2 + h^2 - 20h + 100 = 10^2 = 100$$

or

$$r = \sqrt{(20h - h^2)}.$$



(b) The water level drops at a rate of 0.1cm per hour. At what rate is the radius of water decreasing when the depth is 5cm?

Solution:

- Variables: r, h, t
- We want: dr/dt
- We know $dh/dt = -0.1$

- Equations relating variables $r^2 = 20h - h^2$
- Solving the problem: Differentiating the last equation with respect to t, we get

$$2r\frac{dr}{dt} = 20\frac{dh}{dt} - 2h\frac{dh}{dt}$$

or

$$\frac{dr}{dt} = \frac{10}{r}\frac{dh}{dt} - \frac{h}{r}\frac{dh}{dt}.$$

When the depth is 5, we have $h = 5$ and $r = \sqrt{100-25} = \sqrt{75}$. This gives

$$\frac{dr}{dt} = \frac{10}{\sqrt{75}} \cdot (-0.1) - 10 \cdot (-0.1) = -0.154701.$$

6. A gas station stands at the interesection of a north-south road and an east-west road. A police car is traveling toward the gas station from the east, chasing a stolen truck which is traveling north away from the gas station. The speed of the police car is 100mph at the moment it is 3 miles from the gas station. At the same time , the truck is 4 miles from the gas station going at 80mph. At this moment:

   (a) Is the distance between the between the car and truck increasing or decreasing? How fast? (Distance is measured along a straight line joining the car and truck).

   Solution: This question is almost identical to the one we did in class.

   (b) How does your answer change of the truck is going 70mph instead of 80mph?

   Solution: This question is almost identical to the one we did in class.