

## Section 1.1: Four Ways to Represent a Function

### 1. THE DEFINITION OF A FUNCTION

Functions are one of the most basic tools in mathematics, so we start by considering the definition of a function and all related concepts.

**Definition 1.1.** A function  $f$  is a rule which assigns to each element in a set  $D$ , called the **domain** of  $f$ , exactly one element, called  $f(x)$ , in a set  $R$ .

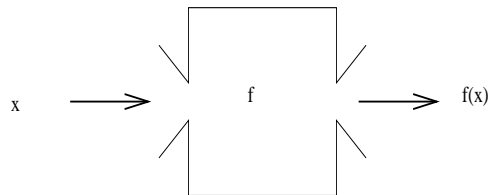
There are a number of important concepts and new terminology related to functions:

- (i) If  $x$  is in the domain of  $f$ , we call  $f(x)$  the value of  $f$  at  $x$ .
- (ii) The **range** of  $f$  is the set of all values that  $f$  can take.
- (iii) An element from the domain of  $f(x)$  is called an **input** of  $f(x)$ .
- (iv) An element from the range of  $f(x)$  is called an **output** of  $f(x)$ .
- (v) A symbol representing an arbitrary element from the domain of  $f$  is called an **independent variable**. We usually use  $x$ .
- (vi) A symbol representing range values of  $f$  is called a **dependent variable** (because it depends upon the independent variable). We usually use  $y$ .

A couple of important and often overlooked facts about functions are the following:

- (i) The range of a function depends upon the domain of the function.
- (ii) Functions do not have to be numerical values or have numerical inputs.

Naively, it is sometimes useful to think of a function as a machine - we put the input in one end of the machine and then the machine produces an output according to the rule given by  $f$ .



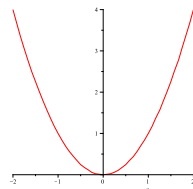
We can also represent a function geometrically. Specifically, if  $f$  is a function, we can plot all the ordered pairs  $(x, f(x))$  on a coordinate plane. We call the set of points  $(x, f(x))$  as  $x$  ranges over all the domain values, the graph of  $f(x)$ . We illustrate our work so far with some examples.

**Example 1.2.** What is the range of  $f(x) = x^2$  if its domain is all  $x \geq 1$ ?

Though the algebraic range of  $f(x) = x^2$  and the corresponding range is  $y \geq 0$ , in this case we are given stronger constraints on the domain, so it is likely that stronger constraints will also exist on the range. Since  $x \geq 1$  and is always positive, it follows that  $x * x \geq 1 * 1 = 1$ , so the range is  $y \geq 1$ .

**Example 1.3.** Sketch the graph of  $f(x) = x^2$ .

For this function, we are plotting all points whose  $y$ -coordinate is the square of the  $x$ -coordinate.



**Example 1.4.** Give an example of function modeling a real life situation where the algebraic domain in the function as a formula is different to the real life constraints.

Consider the area formula for a circle  $A = \pi r^2$ . The algebraic domain for this function is all real numbers. However, a negative value of  $r$ , the radius, does not make sense in this context, so the physical domain is  $r \geq 0$  or the interval  $[0, \infty)$ .

**Example 1.5.** Give an example of a function whose domain and range are not numbers.

There are many different examples - one example is a yes/no survey. The input is the pool of all people taking part in the survey and the range consists of two possible outputs - the answer yes or the answer no.

## 2. FOUR WAYS TO REPRESENT A FUNCTION

We have already talked about different ways to represent a function and you have probably already seen many different ways to represent a function. For this course, we shall considering examples of functions represented in one of four different ways:

- (i) Verbally: A function described using words
- (ii) Numerically: A function described via specific numerical inputs and outputs - usually as a table
- (iii) Visually: A function described using a graph
- (iv) Algebraically: A function described via some specific formula

In traditional calculus and precalculus courses, most functions considered are predominantly represented by formulas and graphs and occasionally numerically. This leads to the obvious question of why we should consider all these different ways of describing functions. The

answer to this lies in how scientific discovery progresses - as scientists trying to model a given phenomenon, we would ideally like some algebraic expression and in order get an algebraic formula, there are a number of different steps usually taken which use all these different representations:

- (i) Identify and verbalize the problem (verbal)
- (ii) Run experiments and collect and write up data (numerical)
- (iii) Use data to sketch a graph (graphical)
- (iv) Use graph to find a formula (algebraic - this is called regression)

To move upward in this list of representations is easy - as scientists however, we want to move downwards, always striving for the perfect algebraic model. Technology has become a wonderful aid in this endeavour.

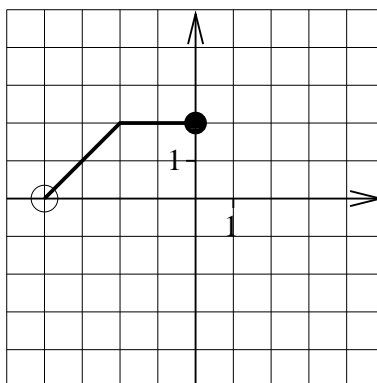
We consider some examples of the different ways to represent a function.

**Example 2.1.** Is the color of a car a function of its make? Why or why not?

If the color of a car is a function of make, that would mean when we input a make into the function, it would output a color. The only way this could be a function is if the car manufacturers made only a single color, which is definitely not the case. Thus this is not a function.

**Example 2.2.** Find the domain and range of the following functions and also the value at  $x = 1$ .

- (i) The graph of  $f(x)$  is given below:



A number  $x$  is in the domain of  $f(x)$  if the point  $(x, y)$  is on the graph of  $f(x)$  for some given  $y$ . Likewise,  $y$  is in the domain of  $f(x)$  if the point  $(x, y)$  is on the graph of  $f(x)$  for some given  $x$ . Thus the domain is  $-2 < x \leq 0$  or  $(-2, 0]$  and the range is  $0 < y \leq 2$  or  $(0, 2]$ .

- (ii)  $f(x) = \sqrt{x+2}$

The domain consists of all real numbers which  $x$  can take. In this case, the only restriction we have is that the square root cannot be negative. Thus we must have  $x + 2 \geq 0$  or  $x \geq -2$ , and in interval notation  $[-2, \infty)$ . The range consists of all outputs. In this case, a square root can never be negative, but can take every other value, so the range is  $y \geq 0$  or  $[0, \infty)$ .

**Example 2.3.** Which of the following define functions?

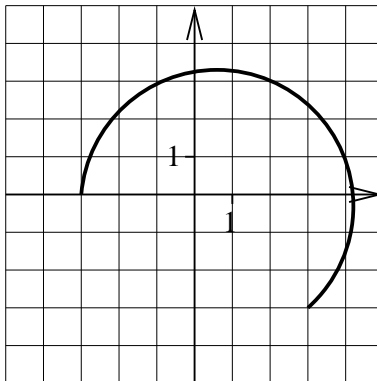
(i)  $y = x^2 - x$

This certainly is a function if we define the independent variable to be  $x$  and the dependent variable to be  $y$ . However, nothing has been stated in the question to suggest that is the case, and it is clear that if  $x$  is the dependent variable, then this does not define a function. Thus, this defines  $y$  as a function of  $x$ , but not  $x$  as a function of  $y$ .

(ii)  $y^2 = x - 1$

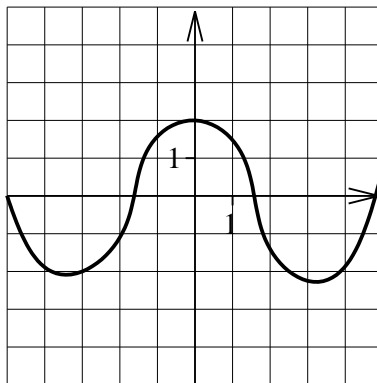
This defines  $x$  as a function of  $y$ , but not  $y$  as a function of  $x$ .

(iii) The graph below:



This cannot possibly define a function since  $x = 4$  has two corresponding  $y$  values, and by definition of a function, any given input has a unique output.

(iv) The graph below:



This cannot define a function since given any  $x$ -value, there is a unique  $y$ -value associated to it (so by definition, it is a function)

Given a graph, there is any easy way geometric way to determine whether it is the graph of a function:

**Result 2.4.** The vertical line test: A graph is the graph of a function if and only if any vertical line touches the graph at most once.

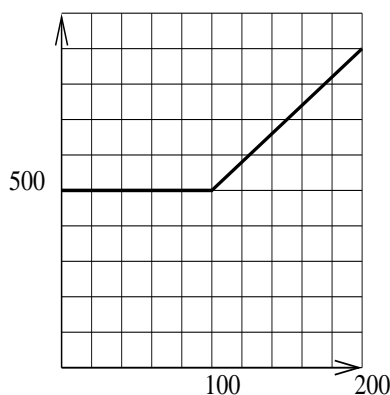
### 3. PIECEWISE DEFINED FUNCTIONS

Sometimes a single formula is not enough to describe a function, and we may require a number of different formulas which depend upon which part of the domain an input comes from. Such functions are called piecewise functions - they give a list of formulas and a list of corresponding domains which determine when the function follows those given formulas. We illustrate with an example.

**Example 3.1.** A company charges a flat rate of \$ 500 to rent up to 100 chairs and every additional chair costs \$ 4. Draw a graph and find a formula for the cost  $C$  as a function of  $n$ , the number of chairs rented.

We need to break the domain up into two pieces where the function has different formulas - when 100 or less chairs are rented, the function is constant with value 500 and when more than 100 are rented, it will be 500 plus 4 times any additional chairs. Thus we get

$$C(n) = \begin{cases} 500 & 0 \leq n \leq 100 \\ 500 + 4(n - 100) & 101 \leq n \end{cases}$$



### 4. OTHER PROPERTIES OF FUNCTIONS

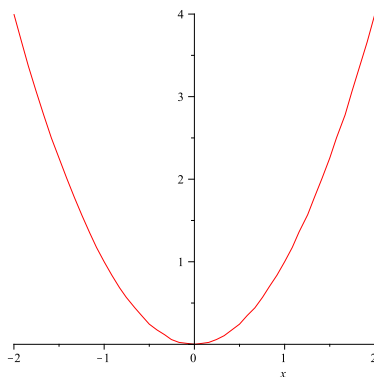
We finish by looking at two other properties of functions - symmetry and increasing/decreasing properties.

**Definition 4.1.** (i) We say a function is even if  $f(-x) = f(x)$  for all  $x$  in the domain of  $f$ . Such a function is symmetric about the  $y$ -axis (reflection symmetry).

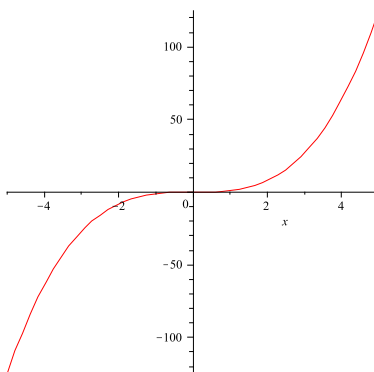
(ii) We say a function is odd if  $f(-x) = -f(x)$  for all  $x$  in the domain of  $f$ . Such a function is symmetric about the origin (rotational symmetry)

Odd and even functions are easy to describe because of their symmetry. We consider a couple of examples.

**Example 4.2.**  $f(x) = x^2$  is even since  $f(-x) = (-x)^2 = x^2 = f(x)$ . Also, we can see this from its graph:



$f(x) = x^3$  is odd since  $f(-x) = (-x)^3 = -x^3 = -f(x)$ . We can also see this from its graph:



One of the important applications of calculus is that it allows us to measure rates of change and determine whether a function is increasing or decreasing. With this in mind, we make formal exactly what we mean by an increasing or decreasing function.

**Definition 4.3.** (i) A function is said to be increasing on an interval  $I$  if for any  $x_1$  and  $x_2$  in  $I$  with  $x_1 < x_2$ , we have  $f(x_1) < f(x_2)$ .

(ii) A function is said to be decreasing on an interval  $I$  if for any  $x_1$  and  $x_2$  in  $I$  with  $x_1 < x_2$ , we have  $f(x_1) > f(x_2)$ .

**Example 4.4.** Looking at the graphs,  $f(x) = x^2$  is decreasing on the interval  $(\infty, 0)$  and increasing on  $(0, \infty)$ , and  $f(x) = x^3$  is increasing on  $(-\infty, \infty)$ .