Section 1.2: A Catalog of Functions

As we discussed in the last section, in the sciences, we often try to find an equation which models some given phenomenon in the real world - for example, temperature as a time of day or velocity in terms of time. Such a model is called a mathematical model. In this section we shall consider some of the more common models used which you should have seen in previous classes.

1. Linear Functions

A linear function is a function whose graph is a line (non-vertical). Equivalently, a linear function is a function with equation

$$f(x) = mx + b$$

where m is called the slope of the function. Such functions are easy to classify - once we know the slope and a single point on the line, or two points on the line, then we can write down an equation for the function. This means that linear models in the real world are arguably the easiest ones to work with. We shall summarize some of the facts about linear functions. The following gives us different ways to write down the equation for a linear function.

Result 1.1. (*i*) (Slope-Intercept form) If a linear function has slope m and y-intercept $(0, b)$, then it has equation $f(x) = mx + b$.

(*ii*) (Point-Slope form) If a linear function has slope m and passed through the point (a, b), then it has equation $f(x) = m(x - a) + b$.

Of these two formulas, point-slope form is probably the most useful since it does not require knowing the intercept (which sometimes we will not be given).

If we are given two points (x_1, y_1) and (x_2, y_2) instead of a point and a slope, we simply calculate the slope by calculating rise over run:

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

We finish by looking at a couple of examples.

Example 1.2. Find a equations for the family of linear functions whose slope is 2.

Since the only condition is that the slope is 2, any such function will be of the form $f(x) = 2x + b$ where b varies over all real numbers.

Example 1.3. Does every line define a linear function?

No - vertical lines do not because they do not pass the vertical line test.

Example 1.4. Find the equation for the linear function which passes through the points $(1, 2)$ and (e, π).

First, the slope is

$$\frac{\Delta y}{\Delta x} = \frac{\pi - 2}{e - 1}.$$

Using point-slope form with the point $(1, 2)$, we get

$$y = \left(\frac{\pi - 2}{e - 1}\right)(x - 1) + 2.$$

Alternatively, we could have had

$$y = \left(\frac{\pi - 2}{e - 1}\right)(x - e) + \pi.$$

2. Power Functions and Polynomials

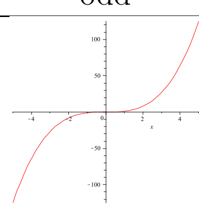
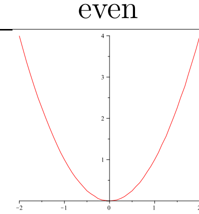
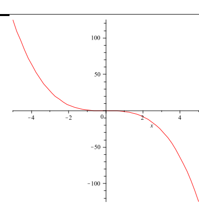
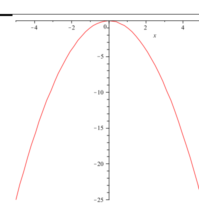
The next simplest type of equation is the polynomial. In order to fully understand polynomials however, we shall first consider power functions.

2.1. **Power Functions.** We start with the definition of a power function.

Definition 2.1. A function of the form $f(x) = ax^n$ for n an integer with $n \geqslant 1$ and a any real number is called a power function.

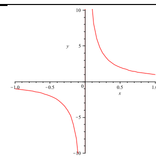
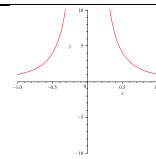
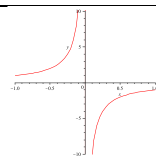
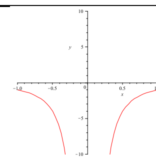
Clearly the domain of all power functions is all real numbers. To determine the ranges, we can look at the different graphs. There are 6 basic shapes for a power function depending upon whether a is positive or negative and whether n is odd, even or 1. Since we have already considered $n = 1$ (linear functions), we instead consider $n \geqslant 2$.

n

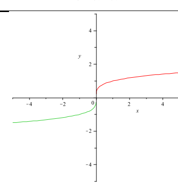
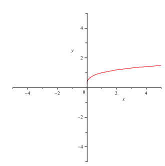
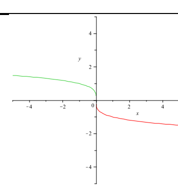
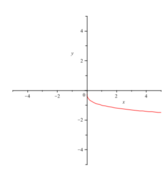
a		odd	even
	positive	 Range: $(-\infty, \infty)$	 Range: $[0, \infty)$
	negative	 Range: $(-\infty, \infty)$	 Range: $(-\infty, 0]$

We can also consider other types of power function where n is a fraction or n is a negative number. These graphs and functions however are more complicated, so we just consider two more explicit examples.

Example 2.2. If we consider negative integers, so power functions of the form $f(x) = ax^n$ where a is any nonzero real number and n is a negative integer, we also get four graphs. In all cases, the domain is all reals except $x = 0$, so the set $(-\infty, 0) \cup (0, \infty)$. The ranges are given in the table below.

		n: odd	n: even
a	positive	 Range: $(-\infty, 0) \cup (0, \infty)$	 Range: $(0, \infty)$
a	negative	 Range: $(-\infty, 0) \cup (0, \infty)$	 Range: $(-\infty, 0)$

Example 2.3. Fractional powers are much more difficult, so instead we just consider power functions of the form $f(x) = ax^{\frac{1}{n}}$ where n is a positive integer (we sometimes call these root functions). Again, we get four different graphs. The main difference in this case however is that the domains are different.

		n: odd	n: even
a	positive	 Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$	 Domain: $(0, \infty)$ Range: $(0, \infty)$
a	negative	 Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$	 Domain: $(0, \infty)$ Range: $(-\infty, 0)$

2.2. **Polynomial Functions.** More general than power functions are polynomials defined as follows:

Definition 2.4. A polynomial is a function with equation $p(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx^n$ where the a_i are integers and $a_n \neq 0$.

For a polynomial as given in the definition, we use the following terminology and have the following facts:

- (i) n is the degree of the polynomial (the largest power with nonzero coefficient).
- (ii) a_nx^n is called the leading term and a_n is called the leading coefficient.
- (iii) The domain is all real numbers.
- (iv) If n is odd, the range is all real numbers. If n is even, and a_n is negative, there is a number N such that the range is $(-\infty, N]$ and if n is even, and a_n is positive, there is a number N such that the range is $[N, \infty)$

One very useful fact says that though we may not be able to completely determine what a polynomial looks like everywhere, we can always work out its end behavior:

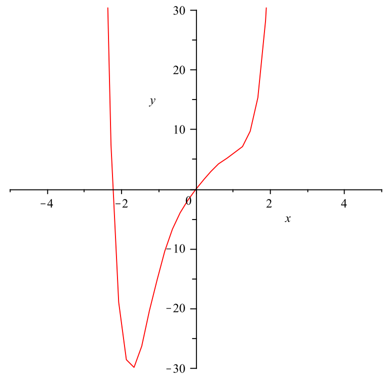
Result 2.5. (The Leading Term Property) Suppose $p(x)$ is a polynomial and a_nx^n is its leading term. Then the end behaviour of $p(x)$ and a_nx^n are the same.

Unlike power functions, we cannot determine all the information about polynomials in general without being given an explicit polynomial. There are some interesting facts which can be derived from just looking at its degree however. We finish with some examples.

Example 2.6. Explain the following facts:

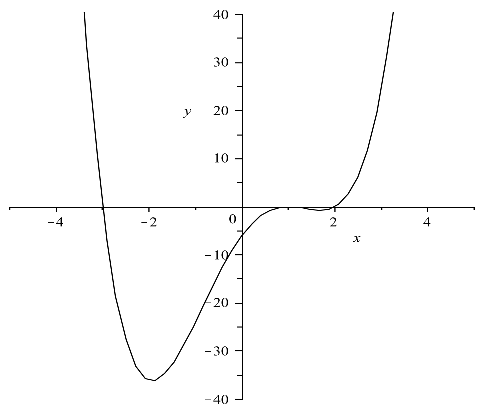
- (i) If the degree of $p(x)$ is n, there are at most n zeros
- (ii) If $p(x)$ has even degree, it can have anywhere between 0 and n zeros
- (iii) If $p(x)$ has odd degree, it can have anywhere between 1 and n zeros
- (iv) If $p(x)$ has even degree, it can have anywhere between 1 and $n - 1$ turning points
- (v) If $p(x)$ has odd degree, it can have anywhere between 0 and $n - 1$ turning points

Example 2.7. How do you know the following is not the complete graph of a degree 9 polynomial? Could it be the complete graph of a degree 6 polynomial?



The end behavior does not match a degree 9 power function. However, it does match the end behavior of a degree 6 power function, so it could conceivably be a degree 6 polynomial.

Example 2.8. Find a possible formula for the polynomial graphed below.



This polynomial has a zero of odd degree at $x = -3$ and $x = 2$ and a zero of even degree at $x = 1$. Therefore a possible formula would be $p(x) = a(x + 3)(x - 1)^2(x - 2)$ where a is some positive real number. Next note that the graph passes through $(0, -6)$, so plugging in those coordinates, we get $a = 1$, so $p(x) = (x + 3)(x - 1)^2(x - 2)$.

3. Rational Functions

Though they are built up from polynomials, rational functions are much more complicated. The formal definition of a rational function is as follows:

Definition 3.1. A rational function is a quotient of polynomials,

$$R(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials. The domain of a rational function are all points where $q(x) \neq 0$.

If a point a is not in the domain of $R(x)$, then $q(a) = 0$ which means that $(x - a)$ divides $q(x)$. If $(x - a)^n$ is the largest power of $x - a$ dividing $q(x)$, we say that a is a zero of $q(x)$ of order n. If a is a zero of $q(x)$ of order n, there are two possible things which can happen at $x = a$:

(i) If $p(x)$ has a smaller degree zero at $x = a$, then there is a vertical asymptote at $x = a$

(ii) If $p(x)$ has an equal or larger degree zero at $x = a$, then there is a hole in the graph at $x = a$.

Similar to polynomials, the end behaviour of a rational function can also be determined by leading coefficients. Specifically, we have the following:

Result 3.2. (The Leading Term Property for Rational Functions) If $R(x) = p(x)/q(x)$ and $a_n x^n$ and $b_m x^m$ are the leading terms of p and q respectively, then the end behavior of $R(x)$ is the same as that of $a_n x^n / b_m x^m$.

Besides these observations, there is little else which can be said about rational functions without looking at explicit examples - not even the range. We shall explore rational functions again later, but we consider an example before we do.

Example 3.3. Find the domain of

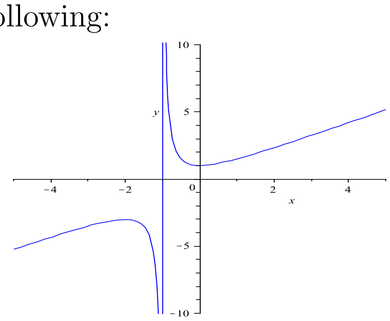
$$R(x) = \frac{x^3 - 1}{x^2 - 1}$$

and sketch its graph.

First, the domain will be all real numbers except the zeros of $x^2 - 1$, so $x \neq \pm 1$. In interval notation, $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. Now notice that $x^3 - 1 = (x - 1)(x^2 + x + 1)$ and $x^2 - 1 = (x - 1)(x + 1$, so the graph of $R(x)$ will be the same as that of

$$\frac{x^2 + x + 1}{x + 1}$$

with a hole at $x = 1$. Finally note that there will be an asymptote at $x = -1$, there are no zeros since the numerator has no zeros, and the end behavior is similar to $x^2/x = x$. Thus the graph must look something like the following:



4. Algebraic Functions

Definition 4.1. An algebraic function is a function which can be constructed from polynomials using the basic algebraic operations (addition, subtraction, multiplication, taking powers and roots).

Algebraic functions are much too numerous to attempt to classify, so instead we give a couple of examples.

Example 4.2. The following are all algebraic functions:

(*i*) $p(x) = (x+1)\sqrt{x}$

(*ii*)

$$q(x) = \frac{\sqrt{x}+1-x}{x^2-\sqrt{x-1}}$$

The graphs of such functions can be almost anything, as can domains/ranges etc.

Example 4.3. Construct an algebraic function with domain $(-2, 2)$.

Consider the function

$$R(x) = \frac{1}{\sqrt{2-x}} + \frac{1}{\sqrt{2+x}}.$$

Clearly we cannot have $x = \pm 2$. Also, since square roots cannot be negative, we must have $2 - x \geqslant 0$ or $2 \geqslant x$ and $2 + x \geqslant 0$ or $x \geqslant -2$, so putting these two together, we have $-2 < x < 2$.

5. Trigonometric Functions

Before we even consider trigonometric functions, we make the very important observation that **degrees do not exist** - in calculus, we always use **radians**.

All trig functions can be built using basic algebraic operations from the two trig functions $\sin(x)$ and $\cos(x)$. Therefore, we shall first recall some basic facts about these two trig functions.

(*i*) The domain of $\sin(x)$ and $\cos(x)$ is all real numbers.

(*ii*) The domains of $\sin(x)$ and $\cos(x)$ are all numbers between -1 and 1 inclusive, so $[-1, 1]$.

(*iii*) $\sin(x)$ and $\cos(x)$ are both periodic with period 2π. This means $\sin(x + 2n\pi = \sin(x))$ and $\cos(x + 2n\pi) = \cos(x)$ for any integer n.

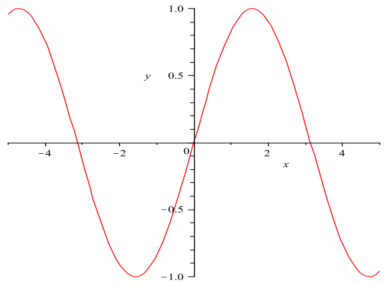
(*iv*) $\sin(x)$ and $\cos(x)$ satisfy the very important identity

$$\sin^2(x) + \cos^2(x) = 1.$$

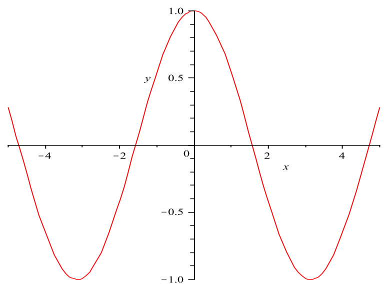
(*v*) The zeros of $\sin(x)$ are at the integer multiples of π.

(*vi*) The zeros of $\cos(x)$ are at the non integer multiples of $\pi/2$.

(*vii*) The graph of $\sin(x)$ is



(*viii*) The graph of $\cos(x)$ is



The other important trigonometric functions are $\tan(x) = \sin(x)/\cos(x)$, $\cot(x) = 1/\tan(x)$, $\sec(x) = 1/\cos(x)$ and $\csc(x) = 1/\sin(x)$.

6. Exponential Functions

Definition 6.1. An exponential function is a function of the form $f(x) = a^x$ where $a > 0$ is some real number.

We postpone a treatment of exponential functions until Section 1.5.

7. Logarithmic Functioins

Definition 7.1. If $f(x) = a^x$ is an exponential function, then the logarithm base a, denoted $\log_a(x)$ is defined to be the inverse function of $f(x)$.

We postpone a treatment of logarithmic functions until Section 1.6.

8. Transcedental Functions

Definition 8.1. We define a transcendental function to ba any function which is not algebraic.

We finish with an example.

Example 8.2. Classify the following functions:

(*i*) $f(x) = x^{\frac{1}{5}}$.
(*ii*) $f(x) = x^9 + x^4$.
(*iii*) $f(x) = \sqrt{1 - x^2}$.
(*iv*) $f(x) = x\sin(x)$.
(*v*) $f(x) = x^x$.

$1-3$ are all algebraic and $4-5$ are transcendental. More specifically, 1 is a power (or root) function and 2 is a polynomial.