# Section 1.3: New Functions from Old

In this section we consider how we can construct new functions from functions we already know via algebraic and other operations.

## 1. Transformations of Functions

The first types of operations we consider on a function are the operations of elementary algebra - multiplication, division, subtraction and addition of a constant to the independent or the dependent variable. We list the different operations and resulting geometric change to the graph. Suppose that $f(x)$ is some function and $c$ is a real number satisfying the conditions outlined below:

- $(i)$ $cf(x)$ is a vertical stretch of the function $f(x)$ by a factor of $c$ for $c > 1$.
- $(ii)$ $cf(x)$ is a vertical compression of the function $f(x)$ by a factor of $1/c$ for $0 < c < 1$.
- $(iii)$ $f(cx)$ is a horizontal compression of the function $f(x)$ by a factor of $c$ for $c > 1$.
- $(iv)$ $f(cx)$ is a horizontal stretch of the function $f(x)$ by a factor of $1/c$ for $0 < c < 1$.
- $(v)$ $-f(x)$ is a reflection of $f(x)$ about the $x$-axis.
- $(vi)$ $f(-x)$ is a reflection of $f(x)$ about the $y$-axis.
- $(vii)$ $f(x) + c$ is an upward vertical shift up by $c$ units for $c > 0$.
- $(viii)$ $f(x) - c$ is a downward vertical shift up by $c$ units for $c > 0$.
- $(ix)$ $f(x + c)$ is a horizontal shift to the left by $c$ units for $c > 0$.
- $(x)$ $f(x - c)$ is a horizontal shift to the right by $c$ units for $c > 0$.

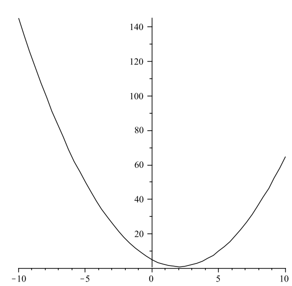
We illustrate with a couple of examples.

**Example 1.1.** Sketch the graph of $y = x^2 - 4x + 5$ by realizing it as a transformation of the function $y = x^2$.

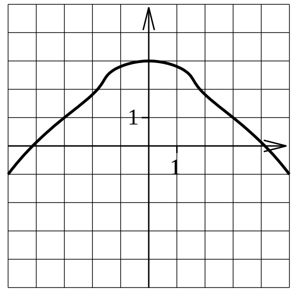
In order to do this, we must first complete the square:

$$y = x^2 - 4x + 5 = (x - 2)^2 - 4 + 5 = (x - 2)^2 + 1.$$

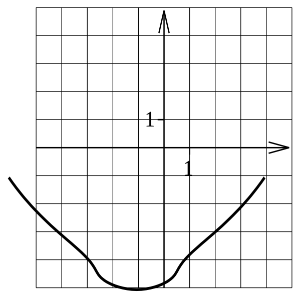
Now we can see that this is simply a shift of $y = x^2$ 2 units to the right and 1 unit up:



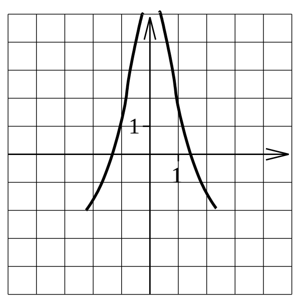
**Example 1.2.** If $f(x)$ is illustrated below, sketch graphs of the following:




$(i)$ -$f(x+1)-2$




$(ii)$ $2f(3x)$




## 2. Combinations of Functions

In this section, we consider how two different functions may be combined using algebraic operations to form a new function. If $f(x)$ and $g(x)$ are functions, there are three basic new functions we can construct:

$(i)$ $(f+g)(x) = f(x) + g(x)$
$(ii)$ $(f-g)(x) = f(x) - g(x)$
$(iii)$ $(f.g)(x) = f(x), g(x)$
$(iv)$ $(f/g)(x) = f(x)/g(x)$

The operations and calculations with such functions are fairly straight forward, though we always have to be careful with domains. We illustrate with a couple of examples.

**Example 2.1.** If $f(x) = \sqrt{x} + 1$ and $g(x) = \sqrt{x}$, find the domain of $(f-g)(x)$.

Observe that $(f-g)(x)=1$ after simplification. This seems to suggest that the domain should be all real numbers. However, we should not forget that $(f-g)(x)$ is a function constructed from $f$ and $g$, so should have the same domain restrictions as $f$ and $g$. Thus the domain will be $x \geqslant 0$ or $(0, \infty)$.

**Example 2.2.** If $f(x)=x^2$ and $g(x)=x$, find the domain of $(f/g)(x)$.

Observe that $(f/g)(x)=x$ after simplification. This seems to suggest that the domain should be all real numbers. However, we should not forget that $(f/g)(x)$ is a function constructed from $f$ and $g$, so should have the same domain restrictions as $f$ and $g$. In particular, we cannot have $g(x)=0$, so the domain will be $x \neq 0$ or $(-\infty, 0) \cup (0, \infty)$.

In general, the domains of the functions given above are the following:

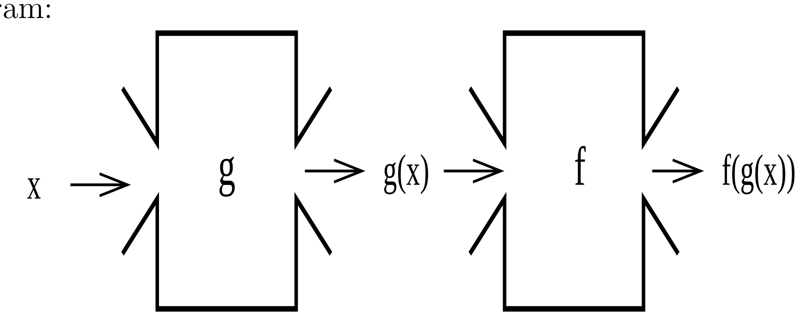
**Result 2.3.** For $(i)-(iii)$, the domains of the combinations of functions will be the domain common to both $f$ and $g$. For $(f/g)(x)$, it will be the domain common to both $f$ and $g$ excluding the zeros of $g$.

## 3. Composition of Functions

The last combination of functions we consider is composition.

**Definition 3.1.** If $f(x)$ and $g(x)$ are functions, we define the composition of $f(x)$ and $g(x)$ denoted $f \circ g(x)$ to be $f(g(x))$ (we input $x$ into $g$ and then input the value $g(x)$ into $f$). The domain consists of all elements in the domain of $g(x)$ whose range lies in the domain of $f$ (so it makes sense to input $g(x)$ into $f$).

If we think of composition in terms of machines, we have the following diagram:




We consider a couple of examples.

**Example 3.2.** Suppose $f(x)=\sqrt{x}$ and $g(x)=x^2$.

$(i)$ Find $f(g(x))$ and specify its domain

We have $f(g(x)) = \sqrt{x^2}$. Now it seems that this should equal $x$, but this cannot be the case since the only outputs of $f(g(x))$ are positive. The domain is clearly all real numbers, and through observation of the function, if we input a number $x$, it squares it and then returns the square root will be equal to

$x$ if $x$ is positive and equal to $-x$ if $x$ is negative. In particular, $f(g(x)) = |x|$ with domain $(-\infty, \infty)$.

$(ii)$ Find $g(f(x))$ and specify its domain

We have $g(f(x)) = (\sqrt{x})^2$. Now it seems that this should equal $x$, but again this cannot be the case since the only inputs of $g(f(x))$ are positive. Through observation of the function, if we input a positive number $x$, it takes the square root and then returns the square which will be equal to $x$. Therefore, it follows that $g(f(x)) = x$ with the domain restriction $[0, \infty)$.

Just as important as being able to compose functions is the skill of decomposing functions, or writing a given function as the composition of other functions. We finish with such an example.

**Example 3.3.** Find functions $f$ and $g$ so that

$$f(g(x)) = \frac{1}{\sqrt{x^2 - 1}}.$$

First we note that there are many choices for $f$ and $g$. We choose $f(x) = 1/\sqrt{x}$ and $g(x) = x^2 - 1$.