# Section 1.5: Exponential Functions

Exponential functions are arguably one of the most important types of functions in the sciences - they can be used to measure population growth, radioactive decay and many other things. In this section we shall formally define an exponential function and explore in detail some of the properties of exponential functions.

## 1. THE DEFINITION OF AN EXPONENTIAL FUNCTION

We start with a formal definition of an exponential function.

**Definition 1.1.** Suppose that $a$ is a positive real number. Then we define the exponential function with base $a$ to be the function $f(x) = a^x$.

This definition seems fairly straight forward. For example, if we consider $f(x) = 2^x$, it is easy to calculate values such as $f(2) = 4$ or $f(3) = 8$, or even $f(-2) = 2^{-2} = 1/4$. A problem does arrive in the definition however when we want to consider non-integral values for $x$ - for example, what do we mean by $f(\pi) = 2^\pi$? We explore this in a little more detail.

Suppose $a$ is some fixed positive number:

(*i*) For a positive integer $n$, as expected, we define

$$a^n = \overbrace{a \cdot a \cdots a}^{n \text{ times}}.$$

(*ii*) For a positive reciprocal of an integer $1/n$, we define

$$a^{\frac{1}{n}} = \sqrt[n]{a},$$

the $n$th root of the number $a$ (which always exists since $a > 0$).

(*iii*) For a positive rational number $p/q$, we define

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

which is defined by the previous two definitions.

(*iv*) Suppose that $x$ is a positive irrational number (we take $\pi$ as an example). Then we define $a^\pi$ as follows:

- We fist observe that $\pi \sim 3.1415\ldots$
- This means that if $a^\pi$ exists, we must have

  $$a^3 < a^\pi < a^4,$$

  and both of these can be calculated since they are integers.
- Next if $a^\pi$ exists, we must have

  $$a^{3.1} < a^\pi < a^{3.2},$$

  and both of these can be calculated since they are rational numbers.

- This means that if $a^{\pi}$ exists, we must have
$$a^{3.14} < a^{\pi} < a^{3.15},$$
and both of these can be calculated since they are rational numbers.
- Continuing this process and observing that these two values get closer and closer together, we can define $a^{\pi}$ as the value these two numbers approach. The same process can be applied for any other irrational number.

($v$) For a negative number $-x$ (so $x$ is positive), we define
$$a^{-x} = \frac{1}{a^x}.$$

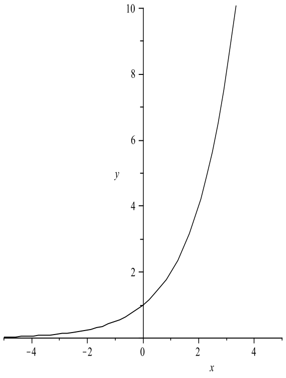
## 2. Basic Properties of Exponential Functions

We list some of the important properties of exponential functions.

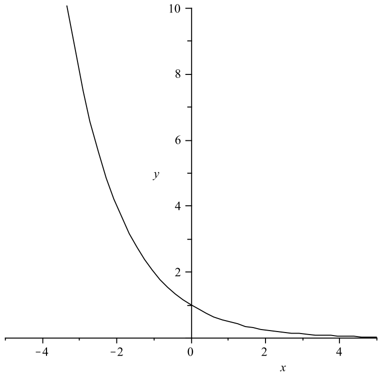
($i$) The domain is all real numbers $(-\infty, \infty)$
($ii$) The range is all positive real numbers, $(0, \infty)$
($iii$) Exponential functions obey the following laws:
  - $a^{x+y} = a^x a^y$
  - $(a^x)^y = a^{xy}$
  - $a^{x-y} = a^x / a^y$
  - $(ab)^x = a^x b^x$

There are two basic types of exponential function - growth (when $a > 1$) and decay (when $0 < a < 1$). We do not consider $a = 1$ since it would just be a constant functions. The graphs for these two different cases look as follows:

Growth, $a > 1$



Decay, $0 < a < 1$

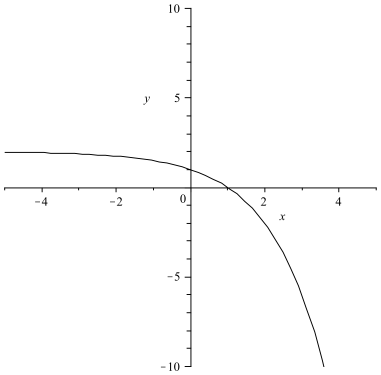


Of course, if $a > 1$, as $a$ gets larger, the graph gets steeper, and if $a < 1$, as $a$ gets smaller, the graph gets steeper.

In addition to regular exponential functions, we shall also be interested in transformations of exponential functions via the basic algebraic operations (which we shall also call exponential functions). We shall also be interested in how they compare to other functions we know. We illustrate with a couple of examples.

**Example 2.1.** Sketch the graph of $y = 2 - (1/2)^{-x}$

This is a reflection about the $y$-axis followed by a reflection about the $x$-axis and then a vertical shift by 2 of the function $y = (1/2)^x$. Therefore, its graph will be:



**Example 2.2.** Which function dominates as $x \to \infty$, $x^{20}$ or $2^x$?

Though $x^2 0$ is a very large function initially, at some point $2^x$ will dominate it - it is a general fact that any exponential growth function will always dominate a polynomial (we shall see why later).

## 3. The Natural Exponential and the Number $e$

The most important base of an exponential function in calculus is the base $e$, and we call the function $f(x) = e^x$ the natural exponential function. It is too early to say exactly why this is the case, but we shall see why later in the course. Before we move on however, we recall how the number $e$ is defined.

**Definition 3.1.** We define the number $e$ as the limit

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

## 4. Applications of Exponential Functions

As we stated at the beginning of the section, exponential functions are very useful in the physical sciences. We finish by illustrating with an example.

**Example 4.1.** Determine a model for a bacteria population in terms of time (measured in hours) if there is an initial population of 2 and a population of 13122 after 8 hours assuming an exponential model.

A general exponential model has the form $P(t) = P_0 a^t$. We need to find $P_0$ and $a$. In order to do this, we plug in the values we know.

First, when $t = 0$, we have $P = 2$, so $2 = P_0 a^0 = P_0$. Then, wen $t = 8$, we have $P = 13122$, so $13122 = 2 * a^8$, or $a^8 = 6561$. Solving for $a$, we get $a = 3$. Thus the model is $P = 2 * 3^t$.