Section 1.6: Inverse Functions and Logarithms

Continuing with the analogy of a function as a machine, it is natural to ask if given the output of some such machine, is there is some other machine we can put this output into and get back the input we placed into the original machine. Equivalently, can we "undo" the function. In this section we consider this problem in detail and examine some specific functions which "undo" functions we are already familiar with.

1. Inverse Functions

We start with a formal definition.

Definition 1.1. Suppose f is a function with domain A and range B. We say f has an inverse function and denote it by $f^{-1}(x)$ with domain B and range A if $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$ for all x.

The first obvious question we need to ask is when an inverse for a given function exists. For this, we need to return to the definition of a functions. In particular, in order to be a function, given any point b in the range of f, in order for f^{-1} to be a function, there must be a single value a in the domain of f so that $f^{-1}(b) = a$ i.e. if there are two values a_1 and a_2 with $f(a_1) = f(a_2) = b$, then there would be two possible outputs for $f^{-1}(b)$ which violates the definition of a function. This motivates the following definition:

Definition 1.2. A function $f\colon A \to B$ is $1-1$ if for every b in B, there exists a unique a in A with $f(a) = b$.

Clearly an inverse function can only exist if a function is $1-1$. Conversely, if a function is $1-1$, then an inverse must exist since it can be defined pointwise. Thus we have the following:

Result 1.3. A function f has an inverse if and only if f is $1-1$. If such a function exists, we denote it by f^{-1}, and its domain is the range of f and its range is the domain of f.

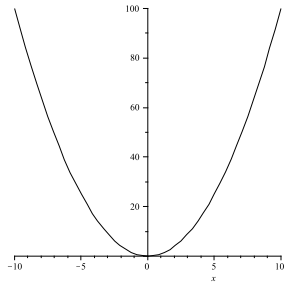
We know that an inverse function exists if and only if a function is $1-1$, so we need a way to determine whether or not a function is $1-1$. There are two ways of doing this:

(i) Geometrically: The Horizontal Line Test - If a function is $1-1$, then every y value gets taken on by f at most once. This means if we draw any horizontal line on the same axis as the graph of f, then it can intersect the graph in *at most* on place (note that it does not have to intersect in any places).

(ii) Algebraically: Plug in two different variables into the equation and set them equal to each other - if the function is $1-1$, after algebraic simplification, you should be able to conclude the two different variables are equal - if not, it is no $1-1$.

Example 1.4. Show that $y = x^2$ is not $1-1$ both algebraically and geometrically.

Geometrically, this is easy to show. Graphing $y = x^2$ on a standard window gives the following graph which clearly does not pass the horizontal line test.



Algebraically, we set $a^2 = b^2$ and solve getting $a = \pm b$. IN particular, a and b are not necessarily equal, so the function is not $1-1$.

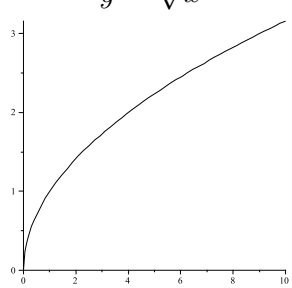
Finding inverse functions is fairly straight forward and only requires basic geometry or algebra.

(i) Geometrically: The graph of the inverse of a $1-1$ function is the reflection of the graph of the original function through the line $y = x$.
(ii) Algebraically: Change y and x in the expression for the function and then solve the equation for y - the resulting answer will be the inverse function.

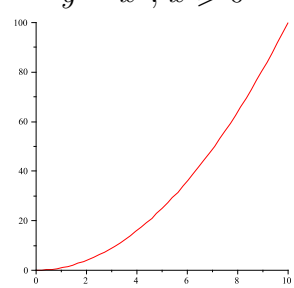
We illustrate.

Example 1.5. (i) Sketch the inverse of $y = \sqrt{x}$ and find a formula for it.

$y = \sqrt{x}$



$y = x^2,\ x \geqslant 0$



Note that the restriction $x \geqslant 0$ on the inverse function is necessary, else the inverse $y = x^2$ would not be $1-1$ (so would not be an inverse function).

(ii) Find the inverse of

$$y = \frac{x-2}{2x+1}.$$

First we switch x and y and then we solve for y:

$$x = \frac{y-2}{2y+1},$$

so

$$2yx + x = y - 2.$$

Solving for y, we have

$$y(2x-1) = 2yx - y = -x - 2,$$

so

$$f^{-1}(x) = y = \frac{-x-2}{2x-1} = \frac{x+2}{1-2x}.$$

2. Special Inverse Functions

There are some functions which are $1-1$ which do not have algebraic inverse functions which we can "solve" for. For such functions, we need to explicitly define them and introduce new terminology.

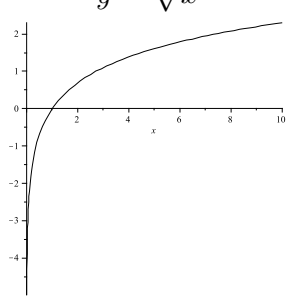
2.1. **Logarithms.** Recall that an exponential function $f(x) = a^x$ is $1-1$ for any $a \neq 1$. This means that an inverse function exists for any exponential function. This motivates the following definition:

Definition 2.1. We define $\log_a(x)$, the log base a of x to be the inverse function of $f(x) = a^x$, the exponential function base a.

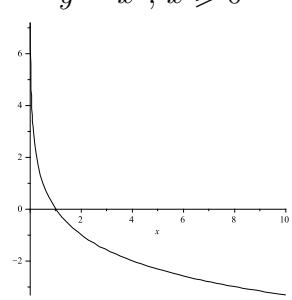
Due to all of the information and properties we know about exponentials, we can transfer this knowledge to find similar information and properties about logarithms:

(i) Properties:
- Domain: $(0, \infty)$, Range: $(-\infty, \infty)$
- Graphs:

$y = \sqrt{x}$



$y = x^2$, $x \geqslant 0$



(ii) Laws of Logarithms
- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(x/y) = \log_a(x) - \log_a(y)$
- $\log_a(x^r) = r \log_a(x)$

Definition 2.2. The Natural Logarithm: We define the natural logarithm denoted $\ln(x)$ to be the logarithm with base e (so the inverse function of the natural exponential function $f(x) = e^x$, or $\log_e(x)$).

We consider some examples.

Example 2.3. Simplify the following:

(*i*) $\log_a(a)$
We have $\log_a(a) = 1$ since $a^1 = a$.

(*ii*) $a^{\log_a(x)}$
We have $a^{\log_a(x)} = x$ since they are inverse functions.

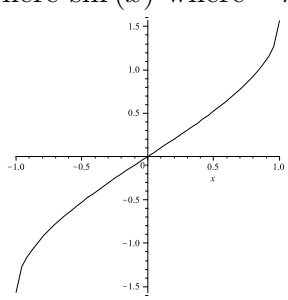
(*iii*) $\log_a(a^x)$
We have $\log_a(a^x) = x$ since they are inverse functions.

(*iv*) $\ln(2e^2)$
Using the logarithm laws, we have

$$\ln(2e^2) = \ln(2) + \ln(e^2) = \ln(2) + 2\ln(e) = \ln(2) + 2.$$

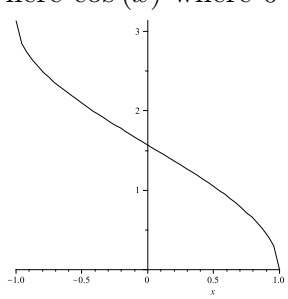
2.2. **Restricted Domains and Inverse Trigonometric Functions.** Other useful types of function are trigonometric functions - they are used to model physical situations where something keeps repeating itself (weather cycles, tides etc.). For all the trigonometric functions however, none of them pass the horizontal line test, so none of them have an inverse function. Under such circumstances, though they have no inverse function on their domains, we can restrict them to a domain where they are $1-1$ and define an inverse on that domain. The standard inverse trigonometric functions are the following:

(*i*) $y = \arcsin(x)$ where $\sin(x)$ where $-\pi/2 \leqslant x \leqslant \pi/2$.



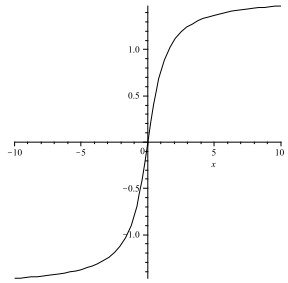
Range: $[-\pi/2, \pi/2]$, Domain: $[-1, 1]$

(*ii*) $y = \arccos(x)$ where $\cos(x)$ where $0 \leqslant x \leqslant \pi$.



Range: $[0, \pi]$, Domain: $[-1, 1]$

(*iii*) $y = \arctan(x)$ where $\tan(x)$ where $-\pi/2 < x < \pi/2$.

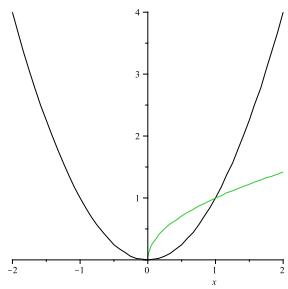


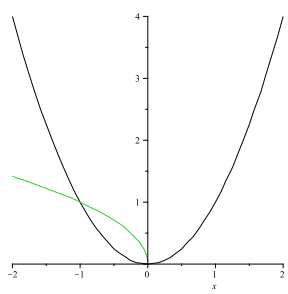
Range: $[-\pi/2, \pi/2]$, Domain: $(-\infty, \infty)$

Of course, there is nothing special about trigonometric functions which allow us to define inverse trigonometric functions on restricted domains - we can do this with any function which is not $1-1$. We illustrate with an example.

Example 2.4. (i) What domain do we need to restrict $y = x^2$ on if it has inverse function $y = \sqrt{x}$ on that domain?

We observe that $y = \sqrt{x}$ is the reflection of $y = x^2$ about the line $y = x$ for $x \geqslant 0$. Thus $y = \sqrt{x}$ is the inverse of $y = x^2$ if we restrict it to the domain $[0, \infty)$.



(ii) What other domain could we restrict $y = x^2$ to and what would its inverse function of that domain?

The other domain on which $y = x^2$ is $1-1$ is the domain $(-\infty, 0]$. On this domain, the inverse function will be $y = \sqrt{-x}$.