

Section 2.1: The Tangent and Velocity Problems

The theory of differential calculus historically stems from two different problems - trying to determine the slope of a tangent line from its equation and trying to find the velocity of a moving object given its position as a function of time. In this section, we shall explore these two problems and explain how a solution to one of these problems is in fact a solution to them both.

1. THE TANGENT PROBLEM

Defining the tangent line to a function at a point is difficult because it requires calculus in order to make a formal definition. Therefore, we start with a naive definition and then develop the theory to make it formal.

Definition 1.1. Suppose $f(x)$ is a function. Then the tangent line to $f(x)$ at $x = a$ is the line which passes through the point $(a, f(a))$ which is the closest linear approximation of $f(x)$ at that point.

With at least a naive definition of tangent line, we can not formally state the Tangent Problem:

Question 1.2. How do we find the equation for the tangent line to $f(x)$ at $x = a$?

Answer. We need a point and the slope.

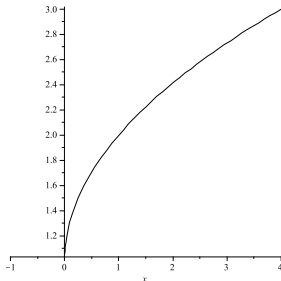
Since we already know the tangent line passes through the point $(a, f(a))$, we are given a point for free. Therefore, the tangent problem translates into the following:

Question 1.3. How do we find the slope of the tangent line to $f(x)$ at $x = a$?

If we solve this problem, then we have solved the tangent problem, so we concentrate on this. We start with an example of how we can do this explicitly.

Example 1.4. Find the equation of the tangent line to $f(x) = \sqrt{x} + 1$ at the point $(1, 2)$.

First we look at the graph of $f(x)$.

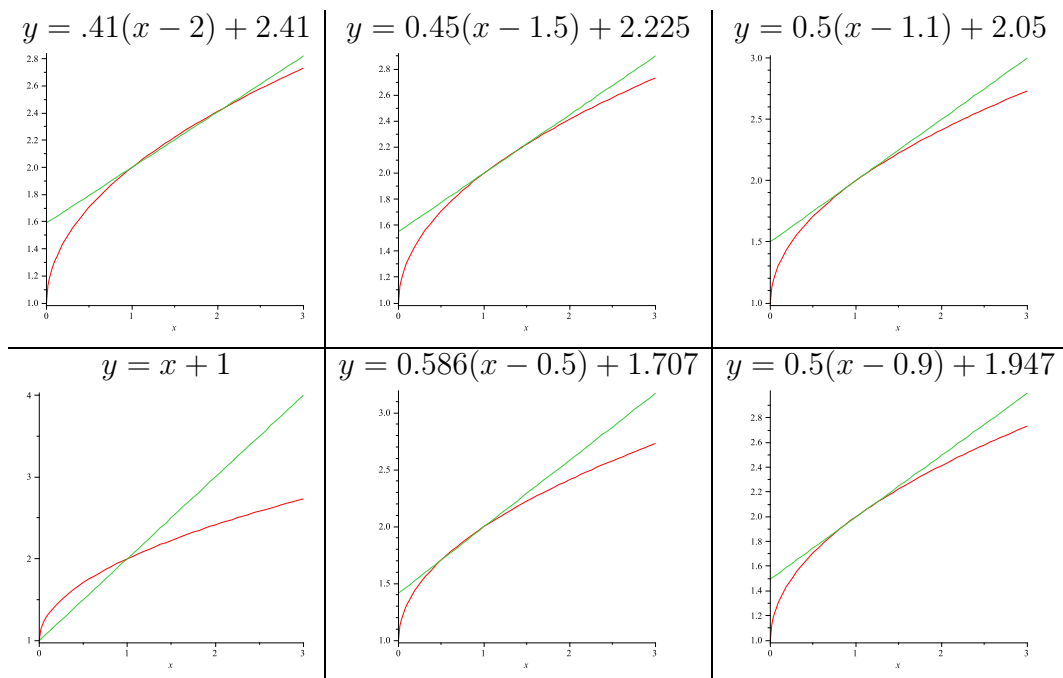


Recall that we can find the equation for a line given any two points on the line. Therefore, we can approximate the tangent line by taking a point (x_0, y_0) satisfying the following:

- It passes through the point $(1, 2)$.
- It passes through a point close to $(1, 2)$ which is also on the graph of $f(x)$.

We call a line passes with the properties above a **secant line**. Below we tabulate the slopes and corresponding equations calculated by taking $\Delta y/\Delta x$ (change in y over change in x) for secant lines and then the graphs of $f(x)$ with each of these secant lines:

x_0	$\Delta x = 1 - x_0$	y_0	$\Delta y = 2 - y_0$	$m = \frac{\Delta y}{\Delta x}$	Equation
2	-1	2.41	-.41	.41	$y = .41(x - 2) + 2.41$
1.5	-0.5	2.225	-.225	.45	$y = 0.45(x - 1.5) + 2.225$
1.1	-0.1	2.05	-0.5	0.05	$y = 0.5(x - 1.1) + 2.05$
0	1	1	1	1	$y = (x - 0) + 1$
0.5	0.5	1.707	0.293	0.586	$y = 0.586(x - 0.5) + 1.707$
0.9	0.1	1.947	0.055	0.5	$y = 0.5(x - 0.9) + 1.947$



Observe that these secant lines are getting closer and closer to what should be the tangent line to $f(x)$ at $x = 1$. Since the equations are getting closer and closer to the equation $l(x) = .5(x - 1) + 2$, it seems that this should be the equation for the tangent line at $(1, 2)$. Thus the slope of $f(x) = \sqrt{x} + 1$ at $x = 1$ is $1/2$ and the equation for the tangent line at that point is $l(x) = .5(x - 1) + 2$.

Our methodology for the previous example seems like it should extend to a general function (though when we do consider the general problem,

we shall have to be careful). Formalizing this, we would have the following:

Result 1.5. The slope of the tangent line $l(x)$ to $f(x)$ at $x = a$ can be calculated as

$$\frac{f(x) - f(a)}{x - a}$$

for values of x very close to a . With this value, we can then solve the tangent line problem.

In order to finish the problem off once and for all, we need to do two things:

- (i) Determine exactly what we mean by “very close to a ” (this will be the focus of Chapter 2)
- (ii) Determine methods to calculate the slope (this will be the focus of Chapter 3)

2. THE VELOCITY PROBLEM

When thought about on a deeper level, the concept of velocity is actually quite complicated. Trying to find a simple answer to the question “what does it mean to say an object is going at 20mph?” is difficult. It is easy to answer a question such as “what does it mean for an object to be traveling at an average speed of 20mph?”, but as soon as we drop the term “average”, and we consider so-called “instantaneous velocity”, we run into problems. Therefore, we shall attempt to define instantaneous velocity ourselves with the use of average velocity. We illustrate with an example.

Example 2.1. Suppose that $f(t) = t^2$ is the position function of a particle on between $0 \leq t \leq 10$. Use the notion of average velocity to determine instantaneous velocity at time $t = 1$.

First, we can estimate the instantaneous velocity using average velocity over an interval close to $t = 1$. In particular, the average velocity for $1 \leq t \leq 2$ will be

$$\frac{2^2 - 1^2}{2 - 1} = 3.$$

Likewise, the average velocity for $0 \leq t \leq 1$ is

$$\frac{1^2 - 0^2}{1 - 0} = 1.$$

Neither of these values measure the actual instantaneous velocity at $t = 1$, but approximate it. The issue however is that the interval we have chosen to approximate it is quite large. Therefore, to get a better approximation to instantaneous velocity, we can choose smaller and smaller intervals:

$$.9 \leq t \leq 1$$

$$\frac{1^2 - (0.9)^2}{1 - .9} = 1.9.$$

$$1 \leq t \leq 1.1$$

$$\frac{(1.1)^2 - 1^2}{1.1 - 1} = 2.1.$$

Continuing this process, we see as we take smaller and smaller intervals over which to calculate the average velocities, the average velocity appears to get closer to 2. Thus it makes sense to define the instantaneous velocity of this particle at $t = 1$ to be 2.

Our example suggests the following method to define and calculate the instantaneous velocity of a moving object:

Result 2.2. Suppose that the position of a moving object at time t is given by the function $s(t)$. Then the instantaneous velocity at time $t = a$ is calculated as

$$\frac{s(x) - s(a)}{x - a}$$

for values of x getting closer and closer to a (but not equal to a).

Observe that this result is the same as when finding the slope of a tangent line at a point. Thus the tangent problem and velocity problem are basically equivalent. Specifically, we have the following:

Result 2.3. Suppose that the position of a moving object at time t is given by the function $s(t)$. Then the instantaneous velocity at time $t = a$ is equal to the slope of the tangent line to the graph of $s(t)$ at $t = a$.