Section 2.2: The Limit of A Function

When trying to solve the tangent and velocity problem, we came to the conclusion that to solve these problems we needed to evaluate

$$\frac{f(x) - f(a)}{x - a}$$

for values of x very close to a. the problem with this however is that we have not really defined precisely what we mean by "values of x very close to a". Therefore, our task for the next few sections is to formalize these concept so we can finally solve the tangent and velocity problems.

1. Limits

We start with a naive definition (we shall make this definition formal in later sections).

Definition 1.1. We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$ as x approaches a equals L" if as we take x values closer and closer to a (from either side), the value of $f(x)$ gets closer and closer to L without taking $x = a$.

Remark 1.2. Limits **do not care** about what happens at the point, only what happens **close to a point**.

We illustrate with an example.

Example 1.3. Evaluate the limit

$$\lim_{x \to 0} \frac{\sin(x)}{x}.$$

First we observe that this function is not defined at $x = 0$. However, by our remark above, this does not mean the limit does not exist since limits do not care about the value at the point, only what happens close by. Therefore, we shall make a table of values of $\sin(x)/x$ for values of x close to 0 (from both sides).

x	$\frac{\sin(x)}{x}$	x	$\frac{\sin(x)}{x}$
-1	0.8415	1	0.8415
$-.5$	0.9589	.5	0.9589
$-.1$	0.9983	.1	0.9983
$-.01$	~ 1	.01	~ 1

Though in principal anything could happen for values of x between $-.1$ and .1, looking at the values, it seems that as x gets closer and closer to 0, $\sin(x)/x$ gets closer and closer to 1. Thus we write

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Note that since $\sin(x)/x$ is an even function, we only need to do the calculations from one side of the equations (the values on the other side will be equal).

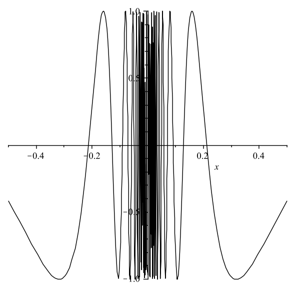
When the value of a function do not approach a single value as x approaches a given number, we say that the limit is "undefined" or "does not exist". There are a number of different scenarios for which a limit may not exist. We describe some through examples.

Example 1.4. (*i*) Explain why the limit

$$\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$$

does not exist.

The graph of $\cos(1/x)$ for values of x close to 1 looks like the following:



In particular, the values of $\cos(1/x)$ as x gets close to 0 are not approaching a single number - in fact they are bouncing between **all** numbers between -1 and 1. Thus the limit does not exists because $\cos(1/x)$ does not approach a single number.

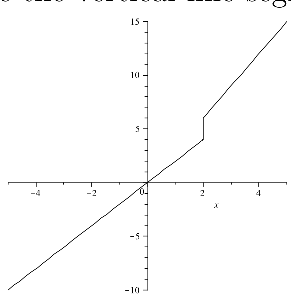
(*ii*) Explain why the limit

$$\lim_{x\to 2} f(x)$$

for

$$f(x) = \begin{cases} 2x & x < 2 \\ 3x & x \geqslant 2 \end{cases}.$$

The graph of this piecewise defined function looks like the following (ignore the vertical line segment):



Though as $x \to 2$ from the left hand side, the function approaches 4 and from the right hand side, it approaches 6, the function does not approach a single value since these two values do not match up. Thus the limit does not exist at this point.

2. One Sided Limits

In the last example we considered, the limits from the left and right hand sides existed but did not match up and so the limit did not exist. Conversely, had these limits matched up, the limit would have existed. We formalize.

Definition 2.1. (*i*) We write

$$\lim_{x \to a^-} f(x) = L$$

and say "the limit of $f(x)$ as x approaches a from the left hand side equals L" if as we take x values closer and closer to a (but with $x < a$), the value of $f(x)$ gets closer and closer to L.

(*ii*) We write

$$\lim_{x \to a^+} f(x) = L$$

and say "the limit of $f(x)$ as x approaches a from the right hand side equals L" if as we take x values closer and closer to a (but with $x > a$), the value of $f(x)$ gets closer and closer to L.

Result 2.2. The limit

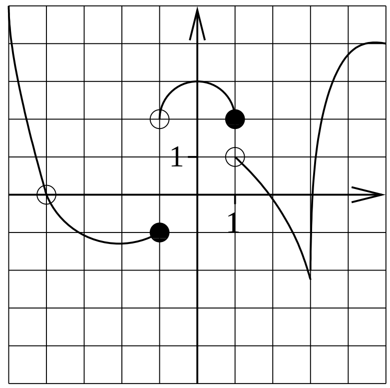
$$\lim_{x \to} f(x)$$

exists if and only if both of the following are true:

(*i*) Both of the one sided limits exist
(*ii*) The one sided limits are equal

This means that in order to determine whether or not a limit exists, it suffices to just concentrate on the two one-sided limits and then compare the answers to check they are equal (provided they exist). We consider some examples.

Example 2.3. Determine the one sided limits at the points $x = -4, -1, 0, 1, 3$ and use your results to determine whether the limit exists at each of these points. Also, calculate $f(x)$ for each of these values of x.



Looking at the graph, we can determine the following:

- $$\lim_{x \to -4^-} f(x) = 0, \lim_{x \to -4^+} f(x) = 0,$$

  so the limit exists at $x = -4$ and

  $$\lim_{x \to -4} f(x) = 0.$$

  Note that $f(4)$ is undefined.

- $$\lim_{x \to -1^-} f(x) = -1, \lim_{x \to -1^+} f(x) = 2,$$

  so the limit does not exist at $x = -1$. Note that $f(-1) = -1$.

- $$\lim_{x \to 0^-} f(x) = 3, \lim_{x \to 0^+} f(x) = 3,$$

  so the limit exists at $x = 0$ and

  $$\lim_{x \to 0} f(x) = 3.$$

  Note that $f(0) = 3$ also.

- $$\lim_{x \to 1^-} f(x) = 3, \lim_{x \to 1^+} f(x) = 1,$$

  so the limit does not exist at $x = 1$. Note that $f(1) = 3$.

- $$\lim_{x \to 3^-} f(x) = -2, \lim_{x \to 3^+} f(x) = -2,$$

  so the exists at $x = 3$ and

  $$\lim_{x \to 3} f(x) = -2.$$

  Note that $f(-2) = -2$.

3. Infinite Limits

If the values of a function $f(x)$ grow without bound as x gets closer to a number a, then using our formal definition of a limit, the limit does not exist at this point. However, unlike the previous examples where left and right hand limits did not match up, or a function infinitely oscillating, even though the limit does not exist if $f(x)$ grows without bound, it does not violate the definition of a limit in the same way - in fact it seems to actually agree with the definition of a limit if we allow infinite limits. Therefore, we define the following:

Definition 3.1. Let $f(x)$ be defined on both sides of a except possibly a. Then

- (i) We write $\lim_{x\to a^-} f(x) = \infty$ if the values of $f(x)$ get larger without bound as we choose values of x closer and closer to a with $x < a$
- (ii) We write $\lim_{x\to a^+} f(x) = \infty$ if the values of $f(x)$ get larger without bound as we choose values of x closer and closer to a with $x > a$
- (iii) We write $\lim_{x\to a^-} f(x) = -\infty$ if the values of $f(x)$ are negative and get larger in absolute value without bound as we choose values of x closer and closer to a with $x < a$
- (iv) We write $\lim_{x\to a^+} f(x) = -\infty$ if the values of $f(x)$ are negative and get larger in absolute value without bound as we choose values of x closer and closer to a with $x > a$

These four definitions can be used to formally define infinite limits as follows:

- (i) We write $\lim_{x\to a} f(x) = \infty$ and say the limits as x goes to a is infinity if $\lim_{x\to a^-} f(x) = \infty$ and $\lim_{x\to a^+} f(x) = \infty$.
- (ii) We write $\lim_{x\to a} f(x) = \infty$ and say the limits as x goes to a is negative infinity if $\lim_{x\to a^-} f(x) = -\infty$ and $\lim_{x\to a^+} f(x) = -\infty$.

Remark 3.2. Note that the graph of a function has a **vertical asymptote** at $x = a$ if and only if there is an infinite limit (either one-sided or two sided) at $x = a$.

We finish with some examples of infinite limits.

Example 3.3. Determine the asymptotes of the following functions and write down their infinite limits.

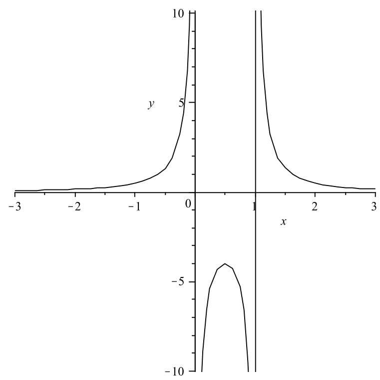
(i)

$$f(x) = \frac{1}{x^2 - x}$$

The function $f(x)$ is undefined at 2 points, $x = 0$ and $x = 1$. To determine the limits at each of these points, we simply plug in numbers close to these points (from either side) and evaluate:

$$\lim_{x\to 0^+} f(x) = -\infty$$
$$\lim_{x\to 0^-} f(x) = \infty$$
$$\lim_{x\to 1^+} f(x) = \infty$$
$$\lim_{x\to 1^-} f(x) = -\infty$$

This information can also be derived from the graph:

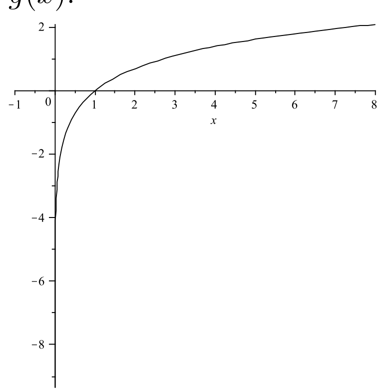


Note that neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 1} f(x) = \infty$ exist since the one sided limits do not match.

(ii)

$$g(x) = \ln(x)$$

First note that $\ln(x)$ is undefined for all $x \leqslant 0$, so the only possible place there could be an asymptote is at $x = 0$. Since $\ln(x)$ is not defined for $x < 0$, the limit $\lim_{x\to 0^-} \ln(x)$ cannot possibly exist. From the positive side, notice that as we input numbers closer and closer to 0 (but larger than 0), the values of $\ln(x)$ are negative and grow in absolute value without bound. Thus $\lim_{x\to 0^+} \ln(x) = -\infty$. We can also see this by looking at the graph of $g(x)$:



(*iii*)

$$h(x) = \sin\left(\frac{1}{x}\right)$$

In this case, even though $1/x$ has one sided infinite limits, the function $\sin(x)$ is bounded between -1 and 1, and hence so will the function $\sin(1/x)$. In particular, the limit at $x = 0$ of $\sin(1/x)$ will not exist, not even as an infinite limit since it infinitely oscillates close to that point. This can also be seen by looking at the graph: