Section 2.3: Calculating Limits using the Limit Laws

In previous sections, we used graphs and numerics to approximate the value of a limit if it exists. The problem with this however is that it does not always give us the correct answer, it may only provide and approximate limit, or even worse, it may suggest a limit exists when in fact it doesn't. Therefore, we need some rules to help evaluate limits of certain common functions and how to evaluate limits under certain algebraic operations.

1. The Basic Limit Laws

We start by listing the basic limit laws.

Result 1.1. Suppose that

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

exist and c is a constant and n is a positive integer. Then the following are true:

(i) Sum Law:

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

(ii) Difference Law:

$$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

(iii) Constant Multiple Rule :

$$\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$$

(iv) Product Law:

$$\lim_{x\to a}(f(x)\times g(x)) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$$

(v) Quotient Law:

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

(vi) Power Law:

$$\lim_{x\to a}(f(x))^n = (\lim_{x\to a} f(x))^n$$

(vii) Root Law:

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

provided if n is even we have

$$\lim_{x\to a} f(x) > 0.$$

In addition to these limit laws, we shall assume the following limits of two simple functions:

Result 1.2. (Limits of Constant and Identity Functions) If c is a constant, the following limits are true:

(i)
$$\lim_{x \to a} c = c$$

(ii)
$$\lim_{x \to a} x = a$$

The limit laws can be used to evaluate limits for many algebraic functions. We illustrate with some examples.

Example 1.3. Evaluate the following limits stating the limit laws used in each step.

(i)
$$\lim_{x \to 2} \frac{x^2 - 2x}{x - 1}$$

$$\lim_{x \to 2} \frac{x^2 - 2x}{x - 1} \overbrace{=}^{\text{Quotient}} \frac{\lim_{x \to 2} x^2 - 2x}{\lim_{x \to 2} x - 1} \overbrace{=}^{\text{Difference}} \frac{\lim_{x \to 2} x^2 - \lim_{x \to 2} 2x}{\lim_{x \to 2} x - \lim_{x \to 2} 1}$$

$$\overbrace{=}^{\text{Power}} \frac{(\lim_{x \to 2} x)^2 - \lim_{x \to 2} 2x}{\lim_{x \to 2} x - \lim_{x \to 2} 1} \overbrace{=}^{\text{Constant Multiple}} \frac{(\lim_{x \to 2} x)^2 - 2\lim_{x \to 2} x}{\lim_{x \to 2} x - \lim_{x \to 2} 1}$$

$$\overbrace{=}^{\text{Constant}} \frac{(\lim_{x \to 2} x)^2 - 2\lim_{x \to 2} x}{(\lim_{x \to 2} x) - 1} \overbrace{=}^{\text{Identity}} \frac{2^2 - 2 * 2}{2 - 1} = 0$$

(ii)
$$\lim_{x \to 8} e^2 + 1$$

$$\lim_{x \to 8} e^2 + 1 \overbrace{=}^{\text{Constant}} e^2 + 1$$

Example 1.4. Show that the limit of a difference of functions may exist even though the individual limits exist.

Consider the functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x}$. Then we have $f(x) - g(x) = 0$ for $x \neq 0$, so the limit exists at $x = 0$ and is equal to 0. However, the limits of the functions $f(x)$ and $g(x)$ do not exist. Note that this tells us that in order to apply the limit laws, the limit of the functions **must** exist - however, just because the limits do not exist, does not mean that the limit of the corresponding combination of functions does not exist.

Our previous results and the previous examples suggest the following result.

Result 1.5. (Direct Substitution) If f is an algebraic function and a is in the domain of $f(x)$, then

$$\lim_{x\to a} f(x) = f(a).$$

In particular, limits of all rational functions and polynomials can be evaluated using direct subsitution.

2. Other Limit Laws

The next example shows a similar result to direct substitution holds in certain special circumstances even though a function may not be defined at a particular point.

Example 2.1. Evaluate

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2}.$$

Observe that

$$\frac{x^2+x-6}{x-2} = \frac{(x-2)(x+3)}{x-2} = x+3$$

for all $x \neq 2$. In particular, if they agree for all different values except at $x = 2$, then they must have the same values close to $x = 2$ and consequently the same limit. Thus

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+3)}{x-2} = \lim_{x\to 2} x+3 = 5$$

by direct subsitution.

Note that in order to find the limit in the previous example, we used the fact that the function agreed with another function at all points except the point where we were evaluating the limit and noting that the limit does not care about what happens at the point. This can be formalized as follows:

Result 2.2. (Replacement Theorem) If $f(x) = g(x)$ for all x in an open interval containing $x = a$ except possible when $x = a$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

provided this limit exists.

We illustrate with a couple of other examples.

Example 2.3. (*i*) Evaluate

$$\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}.$$

We have

$$\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2-2xh-h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} 2x+h = 2x$$

by the replacement Theorem and direct substitution.

(*ii*)

$$\lim_{t\to 0}\frac{1}{t}-\frac{1}{t^2+t}$$

We have

$$\lim_{t\to 0}\frac{1}{t}-\frac{1}{t^2+t}=\lim_{t\to 0}\frac{t^2+t}{t^3+t^2}-\frac{t}{t^3+t^2}=\lim_{t\to 0}\frac{t^2}{t^2(t+1)}=\lim_{t\to 0}\frac{1}{t+1}=1$$

by the replacement Theorem and direct substitution.

In addition to the laws we have already stated, there are a number of other results which can be used to calculate limits. We list them below and then finish with some examples of how they can be applied.

Result 2.4. If $f(x)\leqslant g(x)$ when x is near a (except possibly at a) and then limits of both f and g exist at a, then

$$\lim_{x\to a}f(x)\leqslant\lim_{x\to a}g(x).$$

Result 2.5. (The Squeeze Theorem) If $f(x)\leqslant g(x)\leqslant h(x)$ when x is near a (except possibly at a) and then limits of both h and f exist at a and

$$\lim_{x\to a}f(x)=\lim_{x\to a}h(x)=L$$

then

$$\lim_{x\to a}g(x)=L.$$

Example 2.6. (*i*) Show that

$$\lim_{x\to 0}\frac{|x|}{x}$$

does not exist.

Observe that

$$\lim_{x\to 0^+}\frac{|x|}{x}=1$$

and

$$\lim_{x\to 0^-}\frac{|x|}{x}=-1.$$

Since the left and right hand limits do not match, it follows that the limit does not exist at $x=1$.

(*ii*) Prove that

$$\lim_{x\to 0}x^2\cos\left(\frac{2}{x}\right)=0.$$

Since $-1\leqslant\cos(1/x)\leqslant 1$ for all x, it follows that

$$-x^2\leqslant x^2\cos\left(\frac{2}{x}\right)\leqslant x^2.$$

Next observe that since

$$\lim_{x\to 0} x^2 = \lim_{x\to 0} x^2 = 0$$

by the squeeze theorem, it follows that

$$\lim_{x\to 0} x^2 \cos\left(\frac{2}{x}\right) = 0.$$