Section 2.4: The Precise Definition of a Limit

In the previous sections, we examined the idea of a limit. However, the definitions we gave were more intuitive as opposed to precise. In mathematics, in order to prove results, we cannot base our findings on "intuitive" definitions, but rather formal definitions. Therefore, we need to come up with a precise formal definition of exactly what we mean by a limit.

1. An Example

Before we try to formalize the definition of a limit, we start with an example.

Example 1.1. Suppose that $f(x) = x^2$. Then we know that

$$\lim_{x \to 2} x^2 = 4.$$

With our previous definition of a limit, this means that once x is close enough to 2 we can guarantee that $f(x)$ is within a fixed distance of 4. Therefore, we can consider how close x needs to be to 2 in order to guarantee that $f(x)$ is within 0.1 of 4. Specifically, we want to find the conditions on x which guarantee that

$$|x^2 - 4| < 0.1$$

This means:

$$-0.1 < x^2 - 4 < 0.1$$

so

$$3.9 < x^2 < 4.1$$

and so

$$1.975 < x < 2.025$$

This means for any x in the interval $(1.975, 2.025)$, we can guarantee that $|f(x) - 4| < 0.1$. Alternatively, this means if

$$|x - 2| < 0.025$$

then

$$|f(x) - 4| < 0.1$$

Note that we could repeat this process for any choice of a distance of $f(x)$ from 4.

2. The Formal Definition

We shall now attempt to formalize the definition of a limit. First, we recall again the previous definition we gave for a limit:

Definition 2.1. We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$ as x approaches a equals L" if as we take x values closer and closer to a (from either side), the value of $f(x)$ gets closer and closer to L without taking $x = a$.

We need to make the terms "as we take x values closer and closer to a (from either side)" and "the value of $f(x)$ gets closer and closer to L" more precise. To do this, we make a couple of observations:

- For a fixed a, to guarantee that a number x is within a distance δ of a (from either side), the inequality $|x - a| < \delta$ must be satisfied.
- To say that "the values of $f(x)$ gets closer and closer to L" means that once x is close enough to a, we can guarantee that $f(x)$ is within a fixed distance of L. Specifically, for any fixed distance ε, once x is close enough to a, we can guarantee $|f(x) - L| < \varepsilon$.

Putting these two observations together, and noting our previous example, we get the following definition:

Definition 2.2. We say that the limit of $f(x)$ approaches L as x approaches a and write

$$\lim_{x \to a} f(x) = L$$

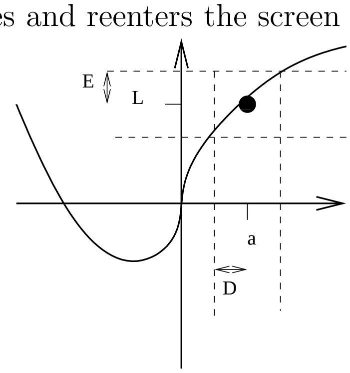
if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x - a| < \delta$ guarantees that $|f(x) - L| < \varepsilon$.

Graphically, if

$$\lim_{x \to a} f(x) = L$$

then for any choice of interval around $y = L$ on the x-axis of radius $\varepsilon > 0$, there exists an interval around $x = a$ of radius $\delta > 0$ such that on the window $x = a - \delta$, $x = a + \delta$, $y = L - \varepsilon$, $y = L + \varepsilon$, the graph of $f(x)$ has the following properties:

- It enters and leaves the screen through the top or bottom of the calculator
- It never leaves and reenters the screen



We illustrate with some examples.

Example 2.3. (*i*) We know that

$$\lim_{x\to 2} f(x) = 5$$

where $f(x) = 2x + 1$. Find the value of δ which guarantees that $|f(x) - 5| < 0.2$.

We want

$$|f(x) - 5| = |(2x + 1) - 5| = |2x - 4| < 0.2.$$

This means

$$-.2 < 2x - 4 < 0.2$$

or

$$3.8 < 2x < 4.2,$$

so

$$1.9 < x < 2.1.$$

Thus we have $-0.1 < x - 2 < 0.1$ or

$$|x - 2| < 0.1.$$

This means that if we choose $\delta = 0.1$, then we can guarantee that $|f(x) - 4| < 0.2$.

(*ii*) Prove that

$$\lim_{x\to 2} f(x) = 5.$$

This is similar to the previous example, but instead of determining a specific value of δ for a fixed value of ε, we need to determine a value of δ for *any* value of ε.

Specifically, we want the value of δ so that

$$|f(x) - 5| = |(2x + 1) - 5| = |2x - 4| < \varepsilon$$

for any chosen value of ε. This means

$$-\varepsilon < 2x - 4 < \varepsilon$$

or

$$4 - \varepsilon < 2x < 4 + \varepsilon,$$

so

$$2 - \frac{\varepsilon}{2} < x < 2 + \frac{\varepsilon}{2}.$$

Thus we have $-\frac{\varepsilon}{2} < x - 2 < \frac{\varepsilon}{2}$ or

$$|x - 2| < \frac{\varepsilon}{2}.$$

This means that if we choose $\delta = \varepsilon/2$, then we can guarantee that $|f(x) - 4| < \varepsilon$ and thus

$$\lim_{x\to 2} f(x) = 5.$$

3. Left Hand Limits, Right Hand Limits and Infinite Limits

Since we have a formal definition for a limit, we can modify it for other types of limit. Specifically, we have the following generalized definitions for one-sided limits.

Definition 3.1. (*i*) We say that the limit of $f(x)$ approaches L as x approaches a from the righta hand side and write

$$\lim_{x\to a^+} f(x) = L$$

if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $a - x < \delta$ guarantees that $|f(x) - L| < \varepsilon$.

(*ii*) We say that the limit of $f(x)$ approaches L as x approaches a from the righta hand side and write

$$\lim_{x\to a^-} f(x) = L$$

if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $x - a < \delta$ guarantees that $|f(x) - L| < \varepsilon$.

We also have the following generalized definitions for infinite limits.

Definition 3.2. (*i*) We say that the limit of $f(x)$ approaches ∞ as x approaches a and write

$$\lim_{x\to a} f(x) = \infty$$

if for any number $M > 0$, there exists a $\delta > 0$ such that $|x - a| < \delta$ guarantees that $f(x) > M$.

(*ii*) We say that the limit of $f(x)$ approaches $-\infty$ as x approaches a and write

$$\lim_{x\to a} f(x) = -\infty$$

if for any number $M < 0$, there exists a $\delta > 0$ such that $|x - a| < \delta$ guarantees that $f(x) < M$.

We consider an example to illustrate how to use these definitions.

Example 3.3. Show that

$$\lim_{x\to 0} -\frac{1}{x^2} = -\infty.$$

Let M be any negative number (so $M = -N$ for some positive number N). Then

$$-\frac{1}{x^2} < M = -N$$

implies

$$N < \frac{1}{x^2}$$

or

$$x^2 < \frac{1}{N}$$

which holds true provided

$$-\frac{1}{\sqrt{N}} < x < \frac{1}{\sqrt{N}}.$$

This means provided

$$|x| < \frac{1}{\sqrt{N}}$$

then

$$-\frac{1}{x^2} < M.$$

Since this works for any value of M, it follows that

$$\lim_{x\to 0} -\frac{1}{x^2} = -\infty.$$

Some additional problems to test your skills:

Example 3.4. (i) Find the value of δ required to guarantee that for $f(x) = 4x - 4$, $|x - 3| < \delta$ implies $|f(x) - 16| < 0.01$.

(ii) Prove that

$$\lim_{x\to 3}(4x + 4) = 16.$$