Section 2.5: Continuity

1. The Definition of Continuity

We start with a naive definition of continuity.

Definition 1.1. We say a function $f(x)$ is continuous if we can draw its graph without lifting out pen off of the paper.

Loosely speaking, this means that a function is continuous provided there are no abrupt changes in the value of the function (the y-values) as the value of x changes. Geometrically speaking, this means that a function is continuous at $x = a$, then as $x \to a$, the function $f(x)$ must get closer to $f(a)$, and in fact to avoid any abrupt changes, it must equal $f(a)$. Thus a formal definition of continuity is the following:

Definition 1.2. A function $f(x)$ is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Before we move on, the very definition of continuity assumes the following facts about $f(x)$:

(*i*) $f(a)$ is defined at $x = a$
(*ii*) $\lim_{x \to a} f(x)$ exists
(*iii*) $\lim_{x \to a} f(x) = f(a)$

If any of these three facts are not true, then $f(x)$ cannot be continuous at $x = a$. We illustrate with some examples of how a function can be discontinuous at a point.

Example 1.3. Explain why the following functions are not continuous at the specified points.

(*i*)

$$f(x) = \frac{x^2 - 1}{x - 1}$$

at $x = 1$.

For this example, note that the limits exists at this point using the replacement Theorem. However, in this case, the function does not exist at this point. We usually call a discontinuity which can be fixed simply by adding a value at a specific point a **removable discontinuity**.

(*ii*)

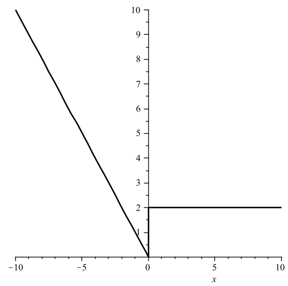
$$f(x) = \frac{1}{x^2 - 4}$$

at $x = 2$.

As with the previous example, this function is not defined at $x = 2$ so certainly could not be continuous at that point. In this case however, the discontinuity is not removable since

$f(x)$ has an infinite limit at that point. We usually call this type of discontinuity an **infinite limit**.

(iii) The piecewise function at $x = 0$ with graph below:



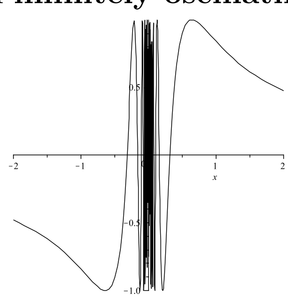
For this example, the function does exist at $x = 0$, but the one sided limits do not match up and hence the limits does not exist at $x = 0$ and so $f(x)$ cannot be continuous at this point. We call a discontinuity where the right and left hand limits exist but do not match up a **jump discontinuity**.

(iv)

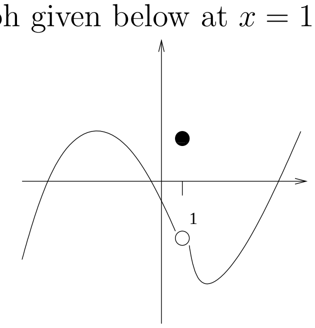
$$g(x) = \sin\left(\frac{1}{x}\right)$$

at $x = 0$.

First observe that the function $g(x)$ is not defined at $x = 0$, so it certainly cannot be continuous at $x = 0$. Next note that it oscillates infinitely between -1 and 1 as $x \to 0$, and thus the limit also does not exist (see graph below). We cal such a discontinuity an **infinitely oscillating** discontinuity.



(v) $g(x)$ with graph given below at $x = 1$.



In this case, both the function and the limit are defined at $x = 1$. However, they are not equal to each other and hence the function is not continuous at $x = 1$. Note that this is a **removable discontinuity**.

As with limits, we can also define the notion of left and right continuity.

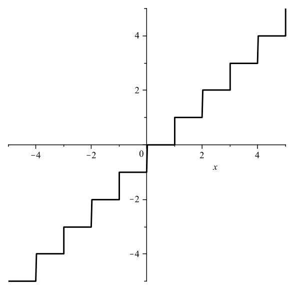
Definition 1.4. (i) A function $f(x)$ is right continuous at $x = a$ if

$$\lim_{x\to a^+} f(x) = f(a).$$

(ii) A function $f(x)$ is left continuous at $x = a$ if

$$\lim_{x\to a^-} f(x) = f(a).$$

Example 1.5. Consider the floor function $f(x) = \lfloor x \rfloor$ (which is defined to be the largest integer less than or equal to x). The graph of $f(x)$ is given below (ignore the vertical lines - they are not part of the graph).



Observe that for any integer n, $f(n) = n$. For any number x between n and $n+1$ with $x \neq n+1$, we have $f(x) = n$. This means that

$$\lim_{x\to n^+} f(x) = n$$

so $f(x)$ is right continuous at each integer n. Next note that for any $n-1 < x < n$, we have $f(x) = n-1$, so since $f(n) = n$, it follows that $f(x)$ is not left continuous at $x = n$. Note that since $f(x)$ is not left continuous at $x = n$ for each integer n, it cannot be continuous at $x = n$. It is continuous everywhere else however.

We have provided a definition for a function being continuous at a point, but usually we are interested in when a function is continuous in an interval. We define such continuity as follows:

Definition 1.6. A function $f(x)$ is said to be continuous on an interval I if it is continuous at every point a in I.

Example 1.7. The floor function $f(x) = \lfloor x \rfloor$ is continuous in every open interval between integers, $(n, n+1)$ for any integer n. However, it is not continuous at any integer n.

2. Examples of Continuous Functions

If we know that a given function is continuous, then we can make certain conclusions about other functions related by algebraic operations. Specifically, we have:

Result 2.1. Suppose that f and g are continuous on an interval I and c is a constant. Then the following are also continuous on I:

(i) $(f + g)(x)$
(ii) $(f - g)(x)$
(iii) $cf(x)$
(iv) $(f \times g)(x)$
(v) $(f/g)(x)$ provided $g(x) \neq 0$

Of the common functions we know, most are continuous. We list them below:

Result 2.2. The following functions are all continuous at all points of there domains:

(i) Polynomials
(ii) Rational Functions
(iii) Root Functions
(iv) Trigonometric Functions
(v) Inverse Trigonometric Functions
(vi) Exponential Functions
(vii) Logarithmic Functions

Finally, we also have the property that the composition of any two continuous functions is also continuous.

Result 2.3. If $g(x)$ is continuous at $x = a$ and $f(x)$ is continuous at $g(a)$, then $f(g(x))$ is continuous at $x = a$.

Summarizing our previous observations, this means that nearly all functions which can be constructed from our original catalog of functions via algebraic operations will be continuous on their domains. We illustrate with an example.

Example 2.4. Determine where the function

$$f(x) = \frac{\arctan(x) - \ln(x)}{2x - 1}$$

is continuous.
Note that this function can be constructed using basic algebraic operations from functions which we already know are continuous everywhere in their domains. Therefore, this function will be continuous everywhere it is defined. Checking the domain, we must have $x \neq 1/2$ and $x > 0$, so $f(x)$ is continuous on the intervals $(0, 1/2) \cup (1/2, \infty)$.

3. Properties of Continuous Functions

Continuous functions have many interesting properties. One of the more interesting properties is the following.

Result 3.1. (Intermediate Value Theorem, IVT) Suppose f is continuous on $[a, b]$. Let N be any number between $f(a)$ and $f(b)$. Then there exists c in $[a, b]$ such that $f(c) = N$.

Summarizing, this theorem basically says that if a function is continuous on an interval and at one end of the interval its y value is $f(a)$ and at the other end, it is $f(b)$, then the function takes every value between $f(a)$ and $f(b)$ on that interval. We illustrate with some examples.

Example 3.2. Explain why an odd degree polynomial $p(x)$ always has a zero.
We know that the end behavior of an odd degree polynomial is opposite, and this any odd degree polynomial must take positive and negative values. Suppose $p(a)$ is negative and $p(b)$ is positive. Then since any polynomial is continuous, the intermediate value Theorem tells us that there exists some number c between a and b with $f(c) = 0$.

Example 3.3. Someone wants to pay you a lot of money to write a program which locates zeros of a continuous function to within a certain degree of accuracy. How can you do this?
We start by testing points, and as soon as we find two different points a and b with $f(a)$ postive and $f(b)$ negative, we know a zero occurs somewhere on the interval $[a, b]$ (or $[b, a]$). For convenience, we shall assume $a < b$. Now we know that the zero occurs somewhere between a and b, but exactly where, we don't know. Therefore to increase the accuracy, we do the following:

- Supposing that $f(a) > 0$ and $f(b) < 0$, we test the sign of $f((a+b)/2)$. If $f((a + b)/2) > 0$, then a zero occurs on the interval $[(a + b)/2, b]$. If $(a + b)/2 < 0$, then a zero occurs on the interval $[a, (a + b)/2]$.

Note that we have halved the size of the interval on which the zero occurs. In order to improve accuracy, we continuous this process until the endpoints of the interval agree within the number of decimal places of accuracy in which we want the zero.

We finish with a couple of examples.

Example 3.4. (*i*) Choose m to make the following function continuous:

$$f(x) = \begin{cases} 2x + m & x < 1 \\ 3x + 2 & x \geqslant 1 \end{cases}$$

The only possible point of discontinuity of this function is at $x = 1$. Therefore, we just need to make sure the two parts of the piecewise equation match up when $x = 1$. That is, we need to choose m such that $2 + m = 5$, or $m = 3$.

(*ii*) Explain how you know

$$f(x) = e^{\frac{1}{x^2+1}}$$

is continuous everywhere.

Note that this function can be constructed using basic algebraic operations from functions which we already know are continuous everywhere in their domains. Therefore, this function will be continuous everywhere it is defined. Since it is defined everywhere, it will be continuous everywhere.