Section 2.7: Derivatives and Rates of Change

Our original motivation to study limits was to help us solve the velocity and tangent problems. In this section, we return to these problems with the tools we have developed which will now allow them to be solved.

1. The Tangent and Velocity Problems

Recall from Section 2.1, we claimed that the slope of the tangent line to $f(x)$ at $x = a$ is equal to

$$\frac{f(x) - f(a)}{x - a}$$

for values of x getting closer and closer to a. Now we have formalized the notion of a limit, we can restate this as follows: the slope of the tangent line to $f(x)$ at $x = a$ is equal to the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided it exists. With this notion of how to determine the slope of a tangent line, we can now make the definition of a tangent line formal.

Definition 1.1. The tangent line to $f(x)$ at $x = a$ is defined to be the line passing through the point $(a, f(a))$ with slope given by the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists (we call this quotient a **difference quotient**).

Remark 1.2. Of course this last statement is necessary - if the limit doesn't exist, then there there must be a problem with defining a tangent line at that point. We shall return to this problem in detail later.

We illustrate with an easy example.

Example 1.3. Determine the equation for the tangent line to $f(x) = x^2$ at $x = 2$.
We have

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2} x + 2 = 4$$

by the replacement theorem. Thus the equation for the tangent line will be

$$l(x) = 4(x - 2) + 4 = 4x - 4.$$

Rather than calculating the slope by using the limit given above, in Calculus, we usually use a different closely related limit defined as follows:

- Suppose that x is a point close to a. Then we define $h = x - a$, so $x = a + h$.

- Notice that as $x \to a$, we have $h \to 0$
- Using this previous observation and substituting h into the limit, we get

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{(a+h)-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Thus we have the following:

Result 1.4. The slope of the tangent line to $f(x)$ ar $x = a$ is equal to the limit

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

provided it exists.

Remark 1.5. The advantage of using this second limit is that we are always considering the limit as $h \to 0$. For the previous limit, we are taking the limit as $x \to a$ where a is conceivably any number.

Example 1.6. Find the equation of the tangent line to $f(x) = \sqrt{x}+1$ at $x = 1$.

$$\lim_{h\to 0} \frac{\sqrt{1+h}+1-(2)}{h} = \lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} \cdot \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$

$$= \lim_{h\to 0} \frac{(1+h)-1}{h(\sqrt{1+h}+1)} = \lim_{h\to 0} \frac{h}{h(\sqrt{1+h}+1)} = \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1} = \frac{1}{2}$$

by direct subsitution. Thus the equation for the tangent line will be

$$l(x) = \frac{1}{2}(x-1)+2.$$

As discussed at the beginning of the chapter, we came to the conclusion that the velocity problem and the tangent problem are in fact equivalent. Thus, we can use our observations above to define instantaneous velocity and provide a formula to calculate it.

Definition 1.7. Suppose an object is moving and its distance relative to some point in terms of time is given by the function $s(t)$. Then the instantaneous velocity at time $t = a$ is defined to be the limit

$$\lim_{h\to 0} \frac{s(a+h)-s(a)}{h}.$$

Example 1.8. The displacement of a particle moving along a line is given by the equation $s(t) = 4t^2 + t$.

(i) Find a general formula for the velocity at time $t = a$.

We have

$$v(a) = \lim_{h\to 0} \frac{4(a+h)^2+(a+h)-(4a^2+a)}{h} = \lim_{h\to 0} \frac{4a^2+8ah+h^2+a+h-4a^2-a}{h}$$

$$= \lim_{h\to 0} \frac{8ah+h^2+h}{h} = \lim_{h\to 0} 8a+h+1 = 8a+1$$

by direct substitution.

(*ii*) Use your formula to find the velocity at $t = 1$, 2 and 8.
We have $v(1) = 9$, $v(2) = 17$ and $v(8) = 65$.

(*iii*) Is the particle accelerating or decelerating?
Since the velocity is increasing, it means the particle is accelerating.

2. Derivatives and Other Rates of Change

The difference quotients we developed in the previous section are so important that we give them their own name.

Definition 2.1. The derivative of a function $f(x)$ at $x = a$ denoted by $f'(a)$ is defined by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists.

Note that the process of determining the derivative is identical to that of determining the slope of a tangent line or the velocity of a moving object, so we shall not repeat any examples for the time being. However, through our previous observations, we can draw the following conclusions:

Result 2.2. (*i*) The derivative $f'(a)$ is equal to the slope of the tangent line to $f(x)$ at $x = a$.

(*ii*) If $s(t)$ is the position of a moving object at time t relative to some point, then its velocity at $t = a$ is equal to the derivative of $s'(a)$.

Of course, there is nothing special about either the tangent problem or the velocity problem - the ideas we have developed can be applied to any situation in which a two quantities are related by a function. Specifically, if y depends upon x, and we know when at x_1, $y = y_1$ and at x_2, $y = y_2$, then the difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

measures the average rate of change of y with respect to x between x_1 and x_2. If we take the limit as $x_2 \to x_1$, then the resulting value can be thought of as the instantaneous rate of change of y with respect to x at the point x_1. Thus we have the following:

Result 2.3. Suppose that the quantity y depends upon the quantity x with relationship described by the function $y = f(x)$. Then the instantaneous rate of change of y with respect to x at $x = a$ is equal to the derivative $f'(a)$.

We finish with some examples.

Example 2.4. Let $f(t)$ be the number of centimeters of water that has fallen since midnight where t is measured in hours. Interpret the following.

(*i*) $f(10) = 3.1$

This means at 10am, 3.1cm of rain has fallen.

(*ii*) $f^{-1}(5) = 6$

This means after 5cm of rain has fallen, 6 hours will have passed since midnight (so it will be 6am).

(*iii*) $f'(10) = 0.4$

This means at 10am, the instantaneous rate of falling water is 0.4cm/hr. Specifically, if the rain does not get any harder or lighter, at 11am, an extra 0.4cm will have fallen.

(*iv*) $(f^{-1})'(5) = 2$

This means that after 5cm have fallen, the the rate of change of time with respect to rainfall is 2hr/cm. Specifically, this means that unless the rain falls at a different rate, it will take 2 more hours for an additional centimeter to fall.

Example 2.5. Suppose that you invest \$100 in stocks and $C(t)$ measures the value of the stock at time t where t is measured in days. Why should you worry if $C'(7) < 0$?
If $C'(7) < 0$, it means that the value of your stock is dropping - you should sell before it loses all its value!