Section 2.8: The Derivative Function

In previous sections we considering the derivative of a function $f(x)$ at a point which was calculated using the limit

$$\lim_{x\to a}\frac{f(a+h)-f(a)}{h}.$$

In this section, instead of considering the derivative at a point, we shall allow a to vary so obtaining instead an expression, or a function, for the derivative at any point which can be evaluated at any point.

1. The Derivative Function

We start with a definition.

Definition 1.1. We define the derivative of $f(x)$ to be the function

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

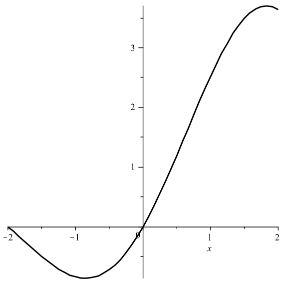
We make a few observations about the derivative:

(i) It is a function (since a limit must always be single valued)
(ii) If we plug in $x=a$ for any a, the result $f'(a)$ will be the derivative of f at $x=a$, or the slope of the tangent line at $x=a$.

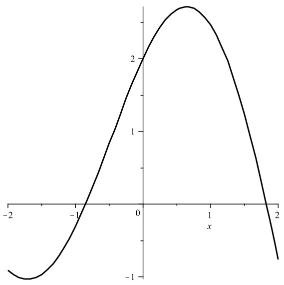
We consider some examples of derivative functions.

Example 1.2. Find the derivative functions of the following.

(i) The functions whose graph is given below:



We observe that the values of the function $f'(x)$ are equal to the slope at any chosen value of x, so the graph of $f'(x)$ will look like the following:



(ii) The function with values given in the table below:

t	0	1	2	3	4	5
d	0	3	8	15	32	89

We shall calculate the values of the derivative by taking the average of the difference quotients of $d(t)$ around each point. First, taking each difference quotient, we have:

$$A_{[1,0]} = \frac{d(1) - d(0)}{1 - 0} = 3, A_{[2,1]} = \frac{d(2) - d(1)}{2 - 1} = 5, A_{[3,2]} = \frac{d(3) - d(2)}{3 - 2} = 7,$$

$$A_{[4,3]} = \frac{d(4) - d(3)}{4 - 3} = 17, A_{[5,4]} = \frac{d(5) - d(4)}{5 - 4} = 57.$$

Then taking the averages (except at the endpoints), we get the following approximate values for the derivative.

t	0	1	2	3	4	5
d'	3	4	6	12	37	57

(iii) The function $f(x) = \sqrt{(x)}$.

Using the difference quotient, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} =$$

$$\lim_{h\to 0} \frac{\sqrt{(x+h)} - \sqrt{(x)}}{h} \cdot \frac{\sqrt{(x+h)} + \sqrt{(x)}}{\sqrt{(x+h)} + \sqrt{(x)}} = \lim_{h\to 0} \frac{x + h - x}{h(\sqrt{(x+h)} + \sqrt{(x)})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{(x+h)} + \sqrt{(x)})} = \lim_{h\to 0} \frac{1}{(\sqrt{(x+h)} + \sqrt{(x)})} = \frac{1}{2\sqrt{x}}$$

Example 1.3. Find the equation of the tangent line to $f(x) = x^2 + x$ at $x = 1$.

First we calculate the derivative using the difference quotient:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h} = \lim_{h\to 0} \frac{2xh + h^2 + h}{h}$$

$$= \lim_{h\to 0} \frac{2x + h + 1}{h} = 2x + 1.$$

Now calculating the derivative at $x = 1$, we have $f'(1) = 3$, so the slope of the tangent line at $x = 1$ is 3. Since $f(1) = 0$, using point slope form, the tangent line has equation

$$l(x) = 3(x - 1).$$

For $y = f(x)$, in addition to the notation $f'(x)$, sometimes the following to denote the derivative function (especially in physics and engineering):

(i)
$$\frac{dy}{dx}$$

(ii)
$$\frac{df}{dx}$$

(iii)
$$Df(x)$$

(iv)
$$D_x f(x)$$

2. Differentiability

We now consider the different possibilities for when a function does not have a derivative at a point. For this, we need the following definition.

Definition 2.1. A function $f(x)$ is said to be differentiable at $x = a$ if $f'(a)$ exists. It is differentiable on an interval (b, c) if it is differentiable at every point in (b, c).

We consider a couple of examples.

Example 2.2. Determine where the following functions are differentiable.

(i) $f(x) = x^2 + x$

From our previous work, we know $f'(x) = 2x + 1$ which is defined everywhere. Thus $f(x)$ is differentiable everywhere.

(ii) $f(x) = \sqrt{(x)}$

From our previous work, we know $f'(x) = 1/2\sqrt{x}$. This is defined only for $x > 0$, and thus $f(x)$ is differentiable provided $x > 0$. In particular, the derivative does not exist at $x = 0$.

Note that a minimal requirement for a function to have a derivative at a point is that it be continuous. Alternatively, if a function is not continuous at a point then there is no way it can be differentiable. Specifically, we have:

Result 2.3. If a function $f(x)$ is differentiable at $x = a$ then it is continuous at $x = a$. Equivalently, if $f(x)$ is not continuous at $x = a$ then it cannot be differentiable at $x = a$.

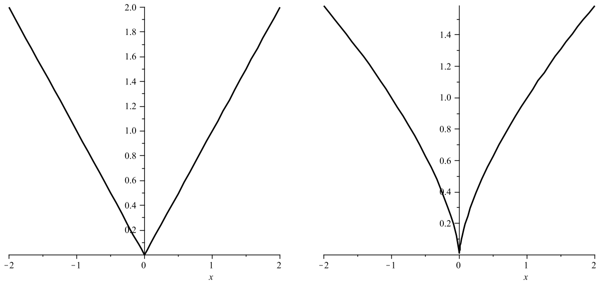
We note that the converse of this statement is **NOT TRUE**. Specifically, we can have a function which is continuous but not differentiable at a point.

Example 2.4. Determine a function which is continuous but not differentiable.
Consider $f(x) = |x|$. Clearly it is continuous, but it is not differentiable since there is a corner at $x = 0$.

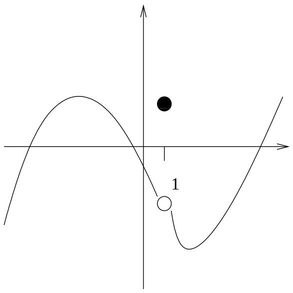
3. NON-DIFFERENTIABLE FUNCTIONS

Our observations in the last section lead us to the obvious question of what other ways a function can fail to be differentiable. We finish with a couple of examples to illustrate how a function may not be differentiable. There are other possibilities, but these are some of the more common ones.

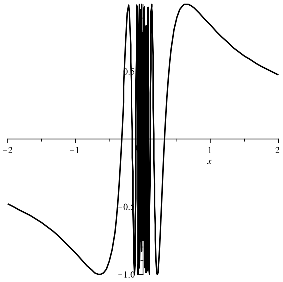
Example 3.1. (*i*) Corners and Cusps



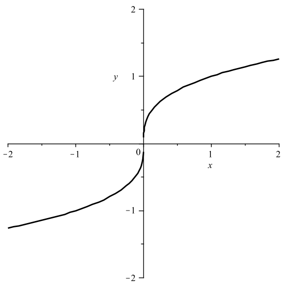
(*ii*) Discontinuities



(*iii*) Infinite Oscillations



(*iv*) Vertical Tangent Lines



4. HIGHER ORDER DERIVATIVES

Since $f'(x)$ is a function, we can also consider its derivative, which we call the second derivative of $f(x)$ and denote by $f''(x)$. Likewise, we can take the derivative of this and in general, we define the nth derivative of a function $f(x)$ to be the function obtained by differentiating n times, and we denote it by $f^{(n)}(x)$ or

$$\frac{d^n y}{dx^n}.$$

In later sections, we shall see some important applications of higher order derivatives. We finish with one example.

Example 4.1. Calculate the 2nd, 3rd and 4th derivatives of $f(x) = x^2 + x$.

From our previous work, we know $f'(x) = 2x + 1$. Calculating the second derivative, we have

$$\begin{aligned} f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} &= \lim_{h\to 0} \frac{2(x+h) + 1 - (2x+1)}{h} \\ = \lim_{h\to 0} \frac{2x + 2h + 1 - 2x - 1}{h} &= \lim_{h\to 0} \frac{2h}{h} = 2. \end{aligned}$$

Calculating t he third derivative, we have

$$f^{(3)}(x) = \lim_{h\to 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h\to 0} \frac{2-2}{h} = 0.$$

Likewise, we can show $f^{(4)}(x) = 0$.