Section 3.1: Derivatives of Polynomials and Exponential Functions

In previous sections we developed the concept of the derivative and derivative function. The only issue with our definition however is that it requires taking the limit of a difference quotient. We have seen that even for easy functions, this can be difficult. In the next few sections, we develop some rules which will allow us to differentiate some of the more common functions. We start with the natural exponential function and polynomials.

1. Derivatives of Power Functions

The easiest type of functions to differentiate are power functions. To obtain a rule for power functions, we start with the easiest - a constant function (or zero power).

Result 1.1. If $f(x) = c$, then $f'(x) = 0$.

Proof.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

□

Next we consider power functions.

Result 1.2. (The Power Rule) Suppose that $f(x) = x^n$ where n is any number. Then $f'(x) = nx^{n-1}$.

Proof. The proof for general values of n is fairly complicated, so we shall just consider the proof for a couple of positive integral values of n (the ideas for higher powers is similar).
When $n = 1$, we have $f(x) = x$. Thus we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x + h - x}{h} = \lim_{h\to 0} \frac{h}{h} = h = 1 * x^0.$$

When $n = 2$, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h} 2x = 2 * x^1.$$

Other values of n follow a similar argument. □

We illustrate with some examples.

Example 1.3. Use the power rule to differentiate the following:

(i) $f(x) = 2$

We have $f(x) = 2 = 2x^0$, so $f'(x) = 0 * 2x^{-1} = 0$.

(*ii*) $f(x) = x^{\pi^e}$

Using the power rule, we have $f'(x) = \pi^e x^{\pi^e - 1}$.

(*iii*) $f(x) = \sqrt{x^3}$

Converting into a power, we have $f(x) = \sqrt{x^3} = x^{3/2}$, so

$$f'(x) = \frac{3x^{1/2}}{2}.$$

(*iv*) $x^{-2/5}$

Using the power rule, we have

$$f'(x) = -\frac{5x^{-7/2}}{2}.$$

Example 1.4. Find the equation of the tangent line to $y = x^{\sqrt{2}}$ at $x = 2$.

First, we use the power rule to find the derivative of $f(x)$. $f'(x) = \sqrt{2}x^{\sqrt{2}-1}$. The derivative of $f(x)$ at $x = 2$ gives us the slope of the tangent line at $x = 2$, so $f'(2) = 2 * 2^{\sqrt{2}-1}$. Using point sloope form, we have

$$y = 2 * 2^{\sqrt{2}-1}(x - 2) + 2^{\sqrt{2}}.$$

2. Derivatives of Polynomials

We would like to transfer our knowledge of power functions to polynomials and other similar functions. However, in order to do this, we need to examine the linear properties of the derivative.

Result 2.1. (Linearity of the Derivative) Suppose $f(x)$ and $g(x)$ are differentiable and c is a constant. Then we have the following:

(*i*) (Constant multiple) If $h(x) = cf(x)$, then $h'(x) = cf'(x)$.
(*ii*) (Sum Rule) If $h(x) = f(x) + g(x)$, then $h'(x) = f'(x) + g'(x)$.
(*iii*) (Difference Rule) If $h(x) = f(x) - g(x)$, then $h'(x) = f'(x) - g'(x)$.

We illustrate why these are true by proving the sum rule.

Proof.

$$h'(x) = \lim_{h \to 0} \frac{h(x+h) - h(x)}{h} = \lim_{h \to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$$

$$= \lim_{h \to 0} \frac{(f(x+h) - f(x)) + (g(x+h) - g(x))}{h}$$

$$= \lim_{h \to 0} \frac{(f(x+h) - f(x))}{h} + \lim_{h \to 0} \frac{(g(x+h) - g(x))}{h} = f'(x) + g'(x)$$

□

Observe that this result now allows us to differentiate any polynomial (and in fact many different algebraic functions). We illustrate with some examples.

Example 2.2. (*i*) If $f(x) = 3x^2 + 2x - 1$, find $f'(x)$

We have $f'(x) = 6x + 2$ using the power rule, the sum rule, the difference rule and the constant multiple rule.

(*ii*) If $f(x) = x^3 - 3x^2 + 3x$, determine where the tangent line is horizontal and write down the equations for tangent lines at these points.

We have $f'(x) = 3x^2 - 6x + 3$. The tangent line is horizontal when $f'(x) = 0$, so $3x^2 - 6x + 3 = 0$. Factoring, we have $3(x^2 - 2x + 1) = 3(x - 1)^2 = 0$, so the only place $f'(x) = 0$ is when $x = 1$. Using point slope form, we get

$$y = 0(x - 1) + 1 = 1.$$

3. Derivatives of Exponential Functions

The last type of basic function we shall consider is the exponential function. In this section we shall only be able to derive a formula for the derivative of the natural exponential function $f(x) = e^x$ - for other exponential functions, we shall come up with a formula which will allow us to determine the derivative in a future section. We start with the general exponential function.

Result 3.1. Suppose $f(x) = a^x$. Then $f'(x) = Ca^x$ where C is the constant

$$C = \lim_{h \to 0} \frac{a^h - 1}{h}.$$

Proof. This question was completed for a homework assignment. For completion, we shall do it again. We have

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} = a^x \lim_{h \to 0} \frac{(a^h - 1)}{h} = Ca^x$$

where

$$C = \lim_{h \to 0} \frac{a^h - 1}{h}.$$

□

This means that if we can evaluate the limit

$$C = \lim_{h \to 0} \frac{a^h - 1}{h}$$

then we can determine the derivative of the general exponential function $f(x) = a^x$. However, this is no easy problem and we will have to return to it in a later section. With the knowledge we currently have though, we can evaluate this limit for the function $f(x) = e^x$.

Result 3.2.
$$\lim_{h\to 0}\frac{e^h-1}{h}=1.$$

Proof. To evaluate this limit, we need to return to the definition of the number e. Recall that
$$e=\lim_{h\to 0}\left(1+h\right)^{\frac{1}{h}}.$$
This means
$$\lim_{h\to 0}e^h=\lim_{h\to 0}1+h$$
or
$$\lim_{h\to 0}(e^h-1)=\lim_{h\to 0}h$$
and so
$$\lim_{h\to 0}\frac{e^h-1}{h}=1.$$
□

Putting these two previous results together, we get the derivative of the natural exponential function. Specifically, we have:

Result 3.3. (Derivative of the Natural Exponential Function) If $f(x)=e^x$, then $f'(x)=e^x$.

We finish with some examples.

Example 3.4. Find the derivatives of the following functions.

(*i*) $f(x)=x+e^x$

Using the sum law, the power law and the exponential law, we have $f'(x)=1+e^x$.

(*ii*) $g(x)=e^x+x^2$

Using the sum law, the power law and the exponential law, we have $f'(x)=2x+e^x$.

(*iii*) $h(x)=e^x+x^e$

Using the sum law, the power law and the exponential law, we have $f'(x)=e^x+ex^{e-1}$.

Example 3.5. (*i*) Write down a general formula for the equation of the tangent line to $f(x)=e^x$ at $x=a$.

First, the derivative of $f(x)=e^x$ is $f'(x)=e^x$. Therefore, the slope at $x=a$ is $f'(a)=e^a$. Therefore, using point-slope form, the equation of the tangent line to $f(x)=e^x$ at (a,e^a) will be
$$y=e^a(x-a)+e^a.$$
.

(*ii*) Use your answer to find the intercept of the equation of a tangent line to $f(x) = e^x$ at $x = a$.

From the previous question, we know the equation of the tangent line. Therefore, by setting this equal to 0, we can determine the x-value of the intercetpt: $0 = e^a(x - a) + e^a$, so $(x-a)+1 = 0$ or $x = 1-a$. Thus the intercept is at the point $(1 - a, 0)$.