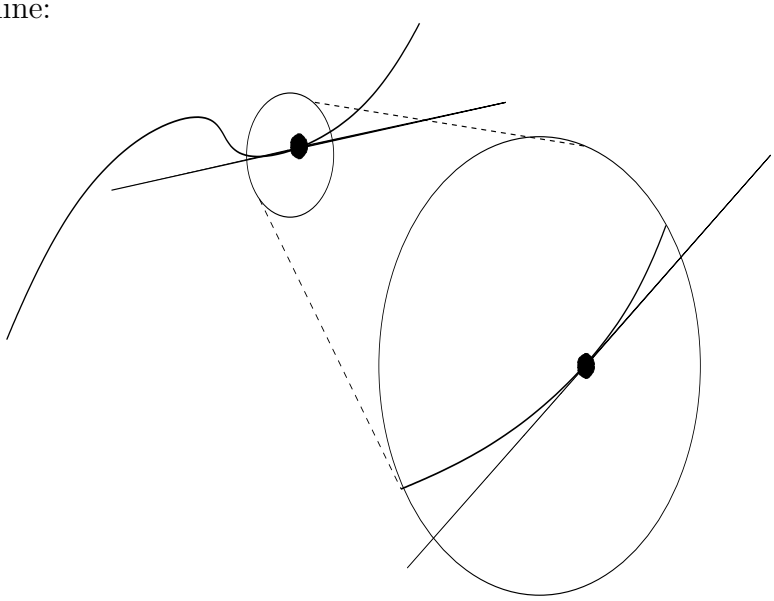
Section 3.10: Linear Approximations and Differentials

In this section, we consider the problem of trying to estimate the value of a function and the value of the change of a function by using the tangent line to a function at a point.

1. Linear Approximations

Suppose that $y = f(x)$ is differentiable at $x = a$. Then as we zoom in closer and closer to the point $(a, f(a))$, $y = f(x)$ looks more and more like a line:



Specifically, as we zoom in to the point $(a, f(a))$, the graph close to this point looks more and more like the tangent line to $y = f(x)$ at $x = a$. In particular, the values of $f(x)$ will be close to the values on the tangent line. For this reason, if $l(x)$ is the tangent line to $f(x)$ at $x = a$, we sometimes call $l(x)$ the linear approximation to $f(x)$ at $x = a$ i.e. it is a linear functions and the values approximate the values of $f(x)$ close to $x = a$ - $f(x) \sim l(x)$ for x close to a. We formalize.

Definition 1.1. We call the tangent line to $f(x)$ at $x = a$, the linear approximation to $f(x)$ at $x = a$. Specifically, the line

$$l(x) = f'(a)(x - a) + f(a)$$

is the linear approximation to $f(x)$ at $x = a$.

The importance of linear approximations is when we are trying to estimate the values of a given function. Specifically, if we are given a very complicated function, instead of using that function to obtain values, we can use a linear approximation. We illustrate with some examples.

Example 1.2. Find the linear approximation to $f(x) = x^x$ at $x = 2$ and use it to approximate the value $2.1^{2.1}$.

Using logarithmic differentiation, we have

$$f'(x) = x^x(\ln(x) + 1)$$

and thus $f'(2) = 4\ln(2) + 4$. This means the linear approximation at $x = 2$ will be

$$l(x) = (4\ln(2) + 4)(x - 2) + 4.$$

Since the linear approximation approximates values of $f(x)$ close to $x = 2$, we have

$$f(2.1) \sim (4\ln(2) + 4)(2.1 - 2) + 4 = 4.677.$$

Note that the calculator answer for this is 4.75 which is quite close to our answer.

Example 1.3. Find the local linearization of

$$f(x) = \sqrt{x^2 + 1}$$

at $x = 1$ and use it to approximate $f(1.1)$. Is you answer an under or overestimate?

We have

$$f'(x) = \frac{x}{x^2 + 1}$$

so

$$f'(1) = \frac{1}{\sqrt{2}}.$$

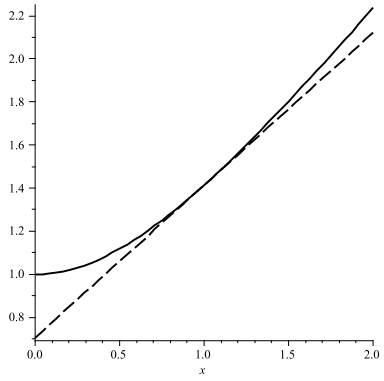
Thus the local linearization at $x = 1$ will be

$$l(x) = \frac{1}{\sqrt{2}}(x - 1) + \sqrt{2}.$$

Close to $x = 1$, we have $l(x) \sim f(x)$ and thus

$$l(1.1) = 1.48.$$

Notice that the actual value is $f(1.1) = 1.49$, so this is an underestimate. This can also be seen by looking at the graph - since it is concave down at that point, the tangent line will be an underestimate at all points.

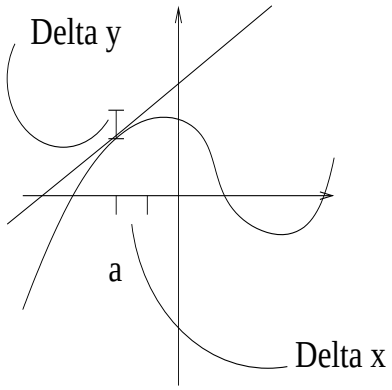


2. Differentials

Another important tool is the differential. Suppose that $f(x)$ is a function which is differentiable at $x = a$. Then the linear approximation at $x = a$ is given by

$$l(x) = f'(a)(x - a) + f(a).$$

Suppose that Δx is some small change in x and suppose we are interested in the corresponding change in y (which we call Δy). Then we can estimate the change by the corresponding change in the tangent line.



Specifically, if we want to approximate the change in $f(x)$ from x to $a + \Delta x$, then we simply calculate the difference in y-values,

$$\Delta y = f(a + \Delta x) - f(a).$$

However, if we cannot calculate $f(a + \Delta x)$, then instead we can approximate it using the difference in y-values on the tangent line

$$\Delta y \simeq l(a+\Delta x)-l(a) = f'(a)(a+\Delta x-a)+f(a)-(f'(a)(a-a)+f(a))$$

$$= f'(a)\Delta x.$$

For very small changes of x, this change will approximate the change in the functions close to $x = a$, and thus we have:

Result 2.1. Suppose $f(x)$ is differentiable at $x = a$ and Δx is a small change in x close to $x = a$. Then the change in $f(x)$ from a to $a + \Delta x$ can be approximated by

$$\Delta y \simeq f'(a)\Delta x.$$

Since the answer becomes more exact as $\Delta x \to 0$, we define the following:

Definition 2.2. We define the differential of $f(x)$ to be

$$dy = f'(x)dx.$$

For a given value of x and dx, the function dy approximates the change in the function $f(x)$ from x to $x + dx$.

Differentials are useful in physics and chemistry when we are measuring very small changes in a quantity. We illustrate with some examples.

Example 2.3. (*i*) Find dy given $y = x^2 - 2x$ when $x = 3$ and $dx = 1/4$.

We have

$$dy = (2x - 2)dx.$$

When $x = 3$ and $dx = 1/4$, we have

$$dy = 4 \cdot \frac{1}{4} = 1.$$

This means $f(x) = x^2 - 2x$ changes by approximately 1 between 3 and $3\frac{1}{4}$. The actual change is

$$(\frac{13}{4}^2 - 2 * \frac{13}{4}) - (3^2 - 2 * 3) = 1.0625.$$

(*ii*) Use a differential to estimate $(8.06)^{2/3}$.

We know $8.06 = 8 + 0.06$, so we can use differentials to approximate the change of $f(x) = x^{2/3}$ at $x = 8$ for $dx = 0.06$. We have

$$dy = \frac{2}{3}x^{-1/3}dx.$$

When $x = 8$ and $dx = 0.06$, we have

$$dy = \frac{2}{3} \cdot 8^{-1/3} \cdot 0.06 = 0.02.$$

This means

$$f(8.06) \simeq 0.02 + f(8) = 0.02 + 8^{2/3} = 4.02.$$

The actual value of is $f(8.06) = 4.019975$, so our approximation is very close.