Section 3.2: The Product and Quotient Laws

In the previous section we determined rules to help us to differentiate the natural exponential function and polynomials. We shall continue developing more rules to differentiate other types of functions in later sections. Before we do this however, we shall develop rules which will allow us to differentiate products and quotients of functions.

1. The Product Rule

When we developed the linear rules for derivatives, we did not state a rule for products. This is simply because the derivative of a product is **not** the derivative of a product. We illustrate with an example.

Example 1.1. Suppose that $f(x) = x^3 = x^2 \cdot x$. Then $f'(x) = 3x^2$. Now note that $f(x) = g(x) \cdot h(x)$ where $g(x) = x^2$ and $h(x) = x$. Then $h'(x) = 1$ and $g'(x) = 2x$, so $g'(x)h'(x) = 2x$. In particular, $f'(x) \neq g'(x)h'(x)$, so the derivative of a product is **not** the

This previous example means we need to derive a rule to calculate the derivative of a product of functions.

Result 1.2. (The Product Rule) Suppose $k(x)$ and $g(x)$ are functions and $f(x) = g(x)h(x)$. Then $f'(x) = g'(x)k(x) + g(x)k'(x)$.

Proof. To show that the rule holds, we need to go back to the original definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{g(x+h)k(x+h) - g(x)k(x)}{h} =$$

$$\lim_{h\to 0} \frac{g(x+h)k(x+h) + g(x+h)k(x) - g(x+h)k(x) - g(x)k(x)}{h} =$$

$$\lim_{h\to 0} \frac{g(x+h)(k(x+h) - k(x)) + k(x)(g(x+h) - g(x))}{h} =$$

$$\lim_{h\to 0} \frac{g(x+h)(k(x+h) - k(x))}{h} + \lim_{h\to 0} \frac{k(x)(g(x+h) - g(x))}{h} =$$

$$g(x) \lim_{h\to 0} \frac{k(x+h) - k(x)}{h} + k(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} =$$

$$g(x)k'(x) + k(x)g'(x).$$

□

We illustrate with some examples.

Example 1.3. Differentiate the following functions using the product rule:

(i) $f(x) = x^3$

We have $f(x) = g(x)k(x)$ where $g(x) = x^2$ and $k(x) = x$. Calculating derivatives, we have

$$\begin{array}{ll} g(x) = x^2 & k(x) = x \\ g'(x) = 2x & k'(x) = 1 \end{array}$$

so

$$f'(x) = g'(x)k(x) + k'(x)g(x) = 2x \cdot x + 1 \cdot x^2 = 3x^2.$$

Note that this matches the answer given by the power rule.

(ii) $f(x) = x^2 e^x$

We have $f(x) = g(x)k(x)$ where

$$\begin{array}{ll} g(x) = x^2 & k(x) = e^x \\ g'(x) = 2x & k'(x) = e^x \end{array}$$

so

$$f'(x) = g'(x)k(x) + k'(x)g(x) = 2xe^x + x^2 e^x.$$

(iii) $f(x) = (3x^2 + 5x)e^x$

We have $f(x) = g(x)k(x)$ where

$$\begin{array}{ll} g(x) = 3x^2 + 5x & k(x) = e^x \\ g'(x) = 6x + 5 & k'(x) = e^x \end{array}$$

so

$$f'(x) = g'(x)k(x) + k'(x)g(x) = (6x + 5)e^x + (3x^2 + 5x)e^x.$$

(iv) $k(x) = e^x/x^2$

This doesn't look like a product, but notice that we can choose $f(x) = g(x)k(x)$ where

$$\begin{array}{ll} g(x) = x^{-2} & k(x) = e^x \\ g'(x) = -2x^{-3} & k'(x) = e^x \end{array}$$

so

$$f'(x) = g'(x)k(x) + k'(x)g(x) = -2x^{-3}e^x + x^{-2}e^x = -\frac{e^x}{2x^3} + \frac{e^x}{x^2}.$$

Example 1.4. If $f(2) = 3$, $f'(2) = 4$, $g(2) = 5$ and $g'(2) = 2$, determine the derivative of $f(x)g(x)$ at $x = 2$.

We have $h(x) = f(x)g(x)$, so calculating derivatives, we have

$$\begin{array}{ll} f(x) & g(x) \\ f'(x) & g'(x) \end{array}$$

so

$$h'(x) = f'(x)g(x) + g'(x)f(x).$$

Thus we have $h'(2) = f'(2)g(2) + g'(2)f(2) = 3 \cdot 2 + 4 \cdot 5 = 6 + 20 = 26.$

Example 1.5. Derive a law to determine the derivative of $(f(x))^2$.

We have $(f(x))^2 = f(x)f(x)$, so calculating derivatives, we have where

$$\begin{array}{cc} f(x) & f(x) \\ f'(x) & f'(x) \end{array}$$

so

$$\frac{d}{dx}(f(x))^2 = f'(x)f(x) + f(x)f'(x) = 2f(x)f'(x).$$

Example 1.6. Differentiate $f(x) = e^{2x}$.

This doesn't look like a product, but notice that we can choose $f(x) = g(x)k(x)$ where

$$\begin{array}{cc} g(x) = e^x & k(x) = e^x \\ g'(x) = e^x & k'(x) = e^x \end{array}$$

so

$$f'(x) = g'(x)k(x) + k'(x)g(x) = e^x e^x + e^x e^x = 2e^x.$$

2. The Quotient Law

Just as we developed a rule to differentiate products of functions, we can develop a rule to differentiate quotients of functions. The proof of the quotient rule is a little more technical than the product rule, so we shall not cover it, and instead simply state it and look at some examples.

Result 2.1. (The Quotient Rule) Suppose

$$f(x) = \frac{g(x)}{k(x)}.$$

Then

$$f'(x) = \frac{g'(x)k(x) - k'(x)g(x)}{(k(x))^2}.$$

Example 2.2. Find the derivatives of the following:

(*i*)

$$q(r) = \frac{3r}{5r+2}$$

We take

$$q(r) = \frac{f(r)}{g(r)}$$

where

$$\begin{array}{ll} f(r) = 3r & g(r) = 5r + 2 \\ f'(r) = 3 & g'(r) = 5r \end{array}$$

Therefore we have

$$q'(r) = \frac{3 \cdot (5r+2) - 5 \cdot 3r}{(5r+2)^2} = \frac{6}{(5r+2)^2}$$

(ii)

$$g(t) = \frac{t-4}{t+4}$$

We take

$$g(t) = \frac{f(t)}{h(t)}$$

where

$$\begin{array}{ll} f(t) = t - 4 & h(t) = t + 4 \\ f'(t) = 1 & h'(t) = 1 \end{array}$$

Therefore we have

$$g'(t) = \frac{(t+4) - (t-4)}{(t+4)^2} = \frac{8}{(t+4)^2}$$

(iii)

$$f(x) = \frac{ax+b}{cx+d}$$

We take

$$f(x) = \frac{g(x)}{h(x)}$$

where

$$\begin{array}{ll} g(x) = ax + b & h(x) = cx + d \\ g'(x) = a & h'(x) = c \end{array}$$

Therefore we have

$$f'(x) = \frac{a(cx+d) - c(ax+b)}{(cx+d)^2} = \frac{ad - bc}{(cx+d)^2}$$

(iv)

$$w(x) = \frac{17e^x}{2x}$$

We take

$$w(x) = \frac{f(x)}{g(x)}$$

where

$$\begin{array}{ll} f(x) = 17e^x & g(x) = 2x \\ f'(x) = 17e^x & g'(x) = 2 \end{array}$$

Therefore we have

$$w'(x) = \frac{34xe^x - 34e^x}{4x^2} = \frac{34e^x(x-1)}{4x^2}$$

(v) $v(t) = e^{-t}$

This doesn't look like a quotient rule problem, but observe that $v(t) = 1/e^t$. Therefore, we take

$$v(t) = \frac{f(t)}{g(t)}$$

where

$$f(t) = 1 \quad g(t) = e^t$$
$$f'(t) = 0 \quad g'(t) = e^t$$

Therefore we have

$$v'(t) = \frac{0 \cdot e^t - e^t}{e^{2t}} = -\frac{1}{e^t} = -e^{-t}.$$

Example 2.3. Derive a rule to differentiate the reciprocal of a function

$$g(x) = \frac{1}{f(x)}.$$

we take

$$g(x) = \frac{h(x)}{f(x)}$$

where

$$h(x) = 1 \quad f(x) = f(x)$$
$$g'(x) = 0 \quad f'(x) = f'(x)$$

Therefore we have

$$h'(t) = \frac{0 \cdot f(x) - f'(x)}{(f(x))^2} = -\frac{f(x)}{(f'(x))^2}.$$