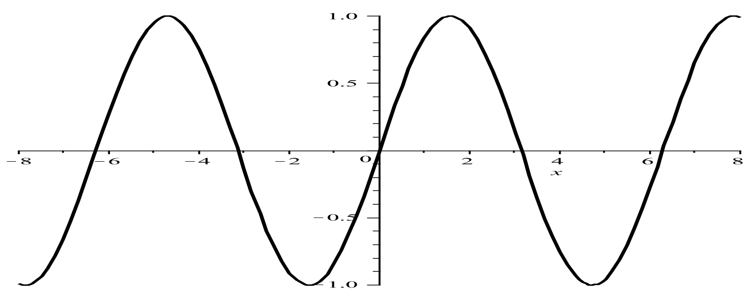
Section 3.3: Derivatives of Trigonometric Functions

In this section we return to the problem of developing rules to differentiate certain common functions. Specifically, our goal for this section is to determine the derivatives of all possible trigonometric functions. Before we start, we note that all such functions can be written as quotients of $\sin(x)$ and $\cos(x)$ and thus we only need determine derivatives of $\sin(x)$ and $\cos(x)$ (all other can be derived using the quotient rule). Since they are both very similar, we shall concentrate on finding the derivative of $\sin(x)$ and then just state the derivative of $\cos(x)$

1. The Derivative of $\sin(x)$ and $\cos(x)$

Before we determine the derivative of $f(x) = \sin(x)$, we make some observations about the properties its derivative should have. First, the graph of $f(x) = \sin(x)$ looks at follows:



By looking at the graph of $f(x) = \sin(x)$, we can derive the following:

- $f'(x)$ will be periodic with period 2π (since the slopes will respeat themselves every 2π units).
- $f'(x) = 0$ when $x = \pi/2$, $3\pi/2$, $5\pi/2$ and at every odd integer multiple of $x = \pi/2$.
- $f'(x)$ takes on its largest value at $x = 0$, $x = 2\pi$ and every even integer multiple of π. The largest value looks approximately equal to 1.
- $f'(x)$ takes on its smallest value at $x = \pi$, $x = 3\pi$ and every odd integer multiple of π. The largest value looks approximately equal to -1.
- All the values of $f'(x)$ are between -1 and 1.

In particular, all of these observations imply that the derivative of $f(x) = \sin(x)$ will be another trigonometric function, and in fact $f'(x) = \cos(x)$ since it is a trigonometric functions which satisfies all these properties. Of course, to be sure that this is indeed the derivative, we need to go back to the definition to check.

Result 1.1. If $f(x) = \sin(x)$ then $f'(x) = \cos(x)$.

Proof. We have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h}$$

$$= \sin(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h}.$$

Therefore, we need to evaluate

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h}$$

and

$$\lim_{h\to 0} \frac{\sin(h)}{h}.$$

First remember that we determined

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$

To determine the other limit, we shall employ one of our limit tricks:

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = \lim_{h\to 0} \left(\frac{\cos(h) - 1}{h}\right)\left(\frac{\cos(h) + 1}{\cos(h) + 1}\right) = \lim_{h\to 0} \frac{\cos^2(h) - 1}{h(\cos(h) + 1)}$$

$$= \lim_{h\to 0} \frac{\sin^2(h)}{h(\cos(h) + 1)} = \left(\lim_{h\to 0} \frac{\sin(h)}{h}\right)\left(\lim_{h\to 0} \frac{\sin(h)}{(\cos(h) + 1)}\right) = 0 \cdot 1 = 0.$$

Thus we have

$$f'(x) = \sin(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} = \cos(x).$$

□

By a similar argument, we can show that if $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$. We summarize:

Result 1.2. (Derivatives of Trigonometric Functions) The following are the derivative of the elementary trigonometric functions:

(*i*) If $f(x) = \sin(x)$ then $f'(x) = \cos(x)$.
(*ii*) If $f(x) = \cos(x)$ then $f'(x) = -\sin(x)$.

2. Examples

We illustrate with some examples.

Example 2.1. Determine the derivatives of the following functions:

(*i*)

$$f(x) = \frac{x}{\cos(x)}$$

This is a direct application of the quotient rule. Specifically, we have:

$$f'(x) = \frac{1 \cdot \cos(x) - x \cdot (-\sin(x))}{\cos^2(x)} = \frac{\cos(x) + x\sin(x)}{\cos^2(x)}$$

(*ii*)

$$f(x) = \cos^3(x) + \sin^2(x)\cos(x)$$

This looks like a complicated product rule problem. However, first we simplify:

$$f(x) = \cos^3(x) + \sin^2(x)\cos(x) = \cos(x)(\cos^2(x) + \sin^2(x)) = \cos(x).$$

Thus $f(x) = \cos(x)$ and so we have $f'(x) = -\sin(x)$.

(*iii*)

$$f(t) = e^t \sin(t)$$

This is a direct application of the product. Specifically, we have:

$$f'(t) = e^t \sin(t) + e^t \cdot (\cos(t)) = e^t \sin(t) + e^t \cos(t).$$

Trigonometric functions are very important in physics and other fields when measuring quantities which follow a repeating pattern (tides, moon cycles etc.). We illustrate with an example.

Example 2.2. A mass on a spring vibrates horizontally on a smooth level surface. Its equation of motion is $x(t) = 8\sin(t)$ where t is measured in seconds and x in centimetres.

(*i*) Find a formula for the velocity of the spring.

The velocity will simply be the derivative of the functions:

$$v(t) = x'(t) = 8\cos(t).$$

(*ii*) Find the position and velocity when $t = \frac{2\pi}{3}$.

We have:

$$\text{Position} = x\left(\frac{2\pi}{3}\right) = 8\sin\left(\frac{2\pi}{3}\right) = \frac{8\sqrt{3}}{2} = 4\sqrt{3}$$

and

$$\text{Velocity} = v\left(\frac{2\pi}{3}\right) = 8\cos\left(\frac{2\pi}{3}\right) = -\frac{8}{2} = -4.$$

Example 2.3. Show that the function $f(x) = \sin(x) + 2x$ has no horizontal tangent lines.

To show that $f(x)$ has no horizontal tangent lines, we need to show that the derivative is never equal to 0. Specifically,

$$f'(x) = \cos(x) + 2.$$

Since $\cos(x) \geqslant -1$, it follows that $f'(x) \geqslant 1$ and thus $f'(x) \neq 0$ for any value of x.

3. Derivatives of other Trigonometric Functions

Since all other trigonometric functions can be built up from $\sin(x)$ and $\cos(x)$ via simple algebraic operations, we can determine their derivatives using the quotient rule.

Result 3.1. (Derivatives of Trigonometric Functions)

(*i*) If $f(x) = \tan(x)$ then $f'(x) = \sec^2(x)$.
(*ii*) If $f(x) = \sec(x)$ then $f'(x) = \sec(x)\tan(x)$.
(*iii*) If $f(x) = \csc(x)$ then $f'(x) = -\csc(x)\cot(x)$.
(*iv*) If $f(x) = \cot(x)$ then $f'(x) = -\csc^2(x)$.

We illustrate with $f(x) = \tan(x)$. Specifically, since

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

we have

$$f'(x) = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$