Section 3.4: The Chain Rule

Suppose that $f(x)$ and $g(x)$ are functions that we know how to differentiate and $h(x) = f(g(x))$. Since $h(x)$ is defined as the composition of f and g, it follows that the derivative of $h(x)$ must somehow be related to the derivatives of f and g. In this section we consider the problem of differentiating functions defined as such (as the composition of two functions).

1. The Statement of the Chain Rule

We start with an example.

Example 1.1. Find the derivative of the function $h(x) = \sqrt{x+1}$.
First note that $h(x)$ is the composition of the functions $g(x) = x+1$ and $f(x) = \sqrt{x+1}$, i.e. $f(g(x)) = h(x)$. This means that the derivative of $h(x)$ must somehow be related to the derivatives of $f(x)$ and $g(x)$. Next observe that the graph of $h(x)$ is the same as that of $f(x)$, except it has been shifted over to the left by a distance of 1 unit. This means that the derivative of $h(x)$ must be the same as that of $f(x)$, just shifted over by 1 unit. Since $f'(x) = 1/2\sqrt{x}$, it follows that $h'(x) = 1/2\sqrt{x+1}$. Notice that this is very similar to the derivative of $f(x)$.

In the previous example we determined the derivative of a composition of functions using geometric arguments. In general, we want a rule to do this. The rule used to differentiate compositions is called the chain rule. Deriving the chain rule is much more difficult that the other rules we have considered, so we shall simply just state it and look at some examples in detail.

Result 1.2. (The Chain Rule) If $f(x)$ and $g(x)$ are differentiable and $h(x) = f(g(x))$, then

$$h'(x) = f'(g(x))g'(x).$$

In words, we say "derivative of the outside function, composed with the inside function multiplied by the derivative of the inside function" where we consider $f(x)$ to be the outside function and $g(x)$ to be the inside function.

Remark 1.3. Note that to apply the chain rule, we need to be good at decomposing functions i.e. writing a function $h(x)$ as a composition of two other functions g and f.

This means to calculate the derivative of a composition of functions, we need to take the following steps:

(i) Decompose $h(x)$ into a product of two functions $f(x)$ and $g(x)$ i.e. $h(x) = f(g(x))$.

(ii) Determine derivatives $f'(x)$ and $g'(x)$ and the composition $f'(g(x))$.

(*iii*) Apply the formula: $h'(x) = f'(g(x))g'(x)$.

We illustrate with some examples.

Example 1.4. Differentiate the following functions.

(*i*) $f(x) = (x^3 + 4x)^{75}$

We have

outside function: $g(x) = x^{75}$; $g'(x) = 75x^{74}$; $g'(h(x)) = 75(x^3 + 4x)^{74}$
inside function: $h(x) = x^3 + 4x$; $h'(x) = 3x^2 + 4$

Therefore, applying the chain rule, we have

$$f'(x) = g'(h(x))h'(x) = 75(x^3 + 4x)^{74}(3x^2 + 4).$$

(*ii*) $g(x) = e^{-kx}$

We have

outside function: $f(x) = e^x$; $f'(x) = e^x$; $f'(h(x)) = e^{-kx}$
inside function: $h(x) = -kx$; $h'(x) = -k$

Therefore, applying the chain rule, we have

$$g'(x) = f'(h(x))h'(x) = -ke^{-kx}.$$

(*iii*) $h(x) = e^{x \cos(x)}$

We have

outside function: $f(x) = e^x$; $f'(x) = e^x$; $f'(g(x)) = e^{x \cos(x)}$
inside function: $g(x) = x \cos(x)$; $g'(x) = \cos(x) - x \sin(x)$

Therefore, applying the chain rule, we have

$$h'(x) = f'(g(x))g'(x) = e^{x \cos(x)}(\cos(x) - x \sin(x)).$$

2. Applications of the Chain Rule

As well as differentiating basic compositions, the chain rule can be used to determine derivatives of functions we cannot yet differentiate as well as determine some new rules. We illustrate with a couple of specific examples.

(*i*) **Differentiating an Exponential Function of Base** a

Suppose $f(x) = a^x$. From previous results, we know that $f(x) = a^x = e^{\ln(a)x}$ and so we can apply the chain rule to differentiate $f(x)$. Specifically, we have

outside function: $g(x) = e^x$; $g'(x) = e^x$; $g'(h(x)) = e^{\ln(a)x} = a^x$
inside function: $h(x) = \ln(a)x$; $h'(x) = \ln(a)$

Therefore, applying the chain rule, we have

$$f'(x) = g'(h(x))h'(x) = \ln(a)a^x.$$

(*ii*) **The General Exponential Rule**

Suppose $h(x) = e^{f(x)}$. We can apply the chain rule to determine a rule which will allow us to differentiate this function and leave the answer in terms of $f(x)$ and its derivative. Specifically, we have

outside function:	$g(x) = e^x$;	$g'(x) = e^x$;	$g'(f(x)) = e^{f(x)}$
inside function:	$f(x)$;	$f'(x)$	

Therefore, applying the chain rule, we have

$$h'(x) = e^{f(x)} f'(x).$$

3. Further Examples

We finish with some further examples of the chain rule.

Example 3.1. Determine a formula to differentiate the composition of three functions $k(x) = f(g(h(x)))$.

We have

outside function:	$f(x)$;	$f'(x)$;	$f'(g(h(x)))$
inside function:	$g(h(x))$;	$g'(h(x))h'(x)$	

Notice that to calculate the derivative of the "inside" function, we needed to apply the chain rule. Now applying the chain rule, we have

$$k'(x) = f'(g(h(x)))g'(h(x))h'(x).$$

Example 3.2. Differentiate

$$k(x) = e^{\sin(e^x)}.$$

We have

outside function:	$f(x) = e^x$;	$f'(x) = e^x$;	$f'(g(x)) = e^{\sin(e^x)}$
inside function:	$g(x) = \sin(e^x)$;	$g'(x) = \cos(e^x)e^x$	

Notice that to calculate the derivative of $g(x)$ we needed to apply the chain rule. Specifically, we have

outside function:	$o(x) = \sin(x)$;	$o'(x) = \cos(x)$;	$o'(i(x)) = \cos(e^x)$
inside function:	$i(x) = e^x$;	$i'(x) = e^x$	

Now applying the chain rule, we have

$$k'(x) = e^{\sin(e^x)} \cos(e^x)e^x.$$

Example 3.3. Some values of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$ are given in the table below.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	2	1	3
1	0	2	1	0
2	1	2	2	1
3	1	3	1	0

Let $h_1(x) = f(g(x))$, $h_2(x) = g(f(x))$, $h_3(x) = f(f(x))$, $h_4(x) = \frac{f(g(x))}{f(x)}$ and $h_5(x) = e^{f(x)}$. Determine the following derivatives:

(i) $h_1'(1)$

We have $h_1'(1) = f'(g(1))g'(1) = 2 \cdot 3 = 6$

(ii) $h_2'(0)$

We have $h_2'(0) = g'(f(0))f'(0) = 0 \cdot 1 = 0$

(iii) $h_3'(2)$

We have $h_3'(2) = f'(f(2))f'(2) = 1 \cdot 2 = 2$

(iv) $h_4'(3)$

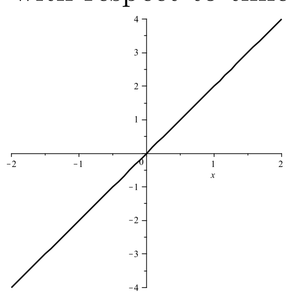
We have

$$h_4'(3) = \frac{f'(g(3))g'(3)g(3) - f(g(3))g'(3)}{(g(3))^2} = \frac{1 \cdot 0 \cdot 3 - 1 \cdot 0}{9} = 0.$$

(v) $h_5'(1)$

We have $h_5'(1) = e^{f(1)}f'(1) = e^0 \cdot 1 = 1.$

Example 3.4. Suppose that the radius of a ball is given as a function of time and the graph is given below. Use this to determine $V'(1)$, the rate of change of volume with respect to time when $t = 2$.



We know that $V(r) = \pi r^2$, and we know that r is a function of t, so it follows that V is a function of t too i.e. $V(r(t))$. Applying the chain rule, it follows that $V'(t) = V'(r(t)) \cdot r'(r) = 2\pi r \cdot r'(t)$. At $t = 1$, looking at the graph, we have $r = 4$ and $r'(t) = 2$ (the slope at that point). Thus we have $V'(2) = 16\pi$.