

Section 3.5: Implicit Differentiation

In the previous sections, we considered the problem of finding the slopes of the tangent line to a given function $y = f(x)$. The idea of a tangent line however is not unique to just functions, but in fact any curve in the plane. For example, circles are not functions, but we can determine the tangent line to a circle at any given point. In this section we consider how to generalize the derivative techniques we have developed and apply them to graphs which are not necessarily graphs of functions.

1. DIFFERENTIATING IMPLICIT FUNCTION

We start with an example.

Example 1.1. Find a formula for the slope of the tangent line to the circle $x^2 + y^2 = 1$ at any point.

Note first that the circle is not the graph of a function, so we need to solve for y and express it as two different functions so we can apply the rules of differentiation. Solving for y we get

$$y = \sqrt{1 - x^2} \text{ or } y = -\sqrt{1 - x^2}.$$

Differentiating, we get

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{1 - x^2}} = -\frac{x}{\sqrt{1 - x^2}} = -\frac{x}{y}$$

or

$$\frac{dy}{dx} = \frac{2x}{2\sqrt{1 - x^2}} = \frac{x}{\sqrt{1 - x^2}} = -\frac{x}{y}.$$

In particular, in both cases, we get

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Our method to calculate the derivative in this example was to first solve the equation for y , then apply the rules of differentiation to the different branches of the functions we get and then put all this information together to write down a derivative. Note that the most important part of this was solving the equation for y meaning we are assuming that y is implicitly defined in the equation as a function of x . Therefore, instead of actually solving the equation for y , we could just treat y as a function of x and then differentiate using the chain rule. We illustrate.

Example 1.2. Find a formula for the slope of the tangent line to the circle $x^2 + y^2 = 1$ at any point.

Assuming that y is defined implicitly as a function of x , so $y = f(x)$ for some function $f(x)$, we have

$$x^2 + (f(x))^2 = 1.$$

Differentiating with respect to x , we get

$$2x + 2f(x)f'(x) = 0.$$

Solving for $f'(x)$, we get

$$f'(x) = -\frac{x}{f(x)} = -\frac{x}{y}.$$

Thus we have

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Note that this is the same as the answer we obtained by solving for y in the previous example.

The method we used to find dy/dx in the previous example is called implicit differentiation. We use implicit differentiation to calculate dy/dx whenever we are given an equation involving the variables x and y which cannot be solved easily for y . However, instead of writing $y = f(x)$, we usually leave the variable as y and write the derivative of y as y' . In general, to implicitly differentiate an equation involving x and y , we do the following:

- (i) Use the sum and difference rules to break up the equation into parts which are no longer sums and differences.
- (ii) For each part, we differentiate with respect to x . In order to do this, we use the standard rules of differentiation. However, if any expression involves the term y , then when that expression is differentiated with respect to x , its derivative must be multiplied by y' (using the chain rule).
- (iii) Put all the terms back into equation form and then solve for y' .

We illustrate with some examples.

Example 1.3. Use implicit differentiation to differentiate the following:

- (i) $x^2 + 2y = y^2 + 1$ and find the slope of the tangent line at $(1, 0)$

Breaking it up into pieces, we need to differentiate x^2 and $2y$ on the left hand side and y^2 and 1 on the right hand side. We examine each one individually. The derivative of x^2 with respect to x will be $2x$, the derivative of $2y$ with respect to x will be $2y'$ using the chain rule. The derivative of y^2 with respect to x will be $2yy'$ using the chain rule, and the derivative of 1 will be 0. Thus we have

$$2x + 2y' = 2yy'.$$

Solving for y' , we get

$$y' = \frac{2x}{2y - 2}.$$

Therefore, the slope of the tangent line at $(1, 0)$ will be

$$\frac{dy}{dx} = \frac{2}{-2} = -1.$$

(ii) $x^2y^2 + x \sin(y) = 4$

Breaking it up into pieces, we need to differentiate x^2y^2 and $x \sin(y)$ on the left hand side and 4 on the right hand side. We examine each one individually. First, to differentiate x^2y^2 , we apply the product rule giving:

$$2xy^2 + 2x^2yy'$$

since the derivative of y^2 will be $2yy'$ using the chain rule. Next, to differentiate $x \sin(y)$, we apply the product rule giving:

$$\sin(y) + x \cos(y)y'$$

since the derivative of $\sin(y)$ will be $\cos(y)y'$ using the chain rule. Finally, the derivative of 4 with respect to x will be 0, so we have

$$2xy^2 + 2x^2yy' + \sin(y) + x \cos(y)y' = 0.$$

Solving for y' , we get

$$y' = \frac{-2xy^2 - \sin(y)}{2x^2y + x \cos(y)}.$$

(iii)

$$e^{\frac{1}{x^2+y^2}} = e$$

Breaking it up into pieces, we need to differentiate

$$e^{\frac{1}{x^2+y^2}}$$

and e . Clearly the derivative of e is 0. To differentiate

$$e^{\frac{1}{x^2+y^2}}$$

we need the chain rule. We have

$$\begin{aligned} \text{outside function: } & g(x) = e^x; & g'(x) &= e^x; & g'(h(x)) &= e^{\frac{1}{x^2+y^2}} \\ \text{inside function: } & h(x) = \frac{1}{x^2+y^2}; & h'(x) &= -\frac{(2x+2yy')}{(x^2+y^2)^2} \end{aligned}$$

Therefore, applying the chain rule, the derivative of

$$e^{\frac{1}{x^2+y^2}}$$

will be

$$-\frac{(2x + 2yy')}{(x^2 + y^2)^2} e^{\frac{1}{x^2+y^2}}$$

Thus we have

$$-\frac{(2x + 2yy')}{(x^2 + y^2)^2} e^{\frac{1}{x^2+y^2}} = 0.$$

Solving for y' , we get

$$y' = -\frac{x}{y}.$$

Observe that we could have simplified algebraically first and we would have obtained the same answer. Specifically,

$$e^{\frac{1}{x^2+y^2}} = e$$

means

$$\frac{1}{x^2 + y^2} = \ln(e) = 1,$$

so $x^2 + y^2 = 1$. From our previous example, we know

$$\frac{dy}{dx} = -\frac{x}{y}$$

for $x^2 + y^2 = 1$.

2. APPLICATIONS OF IMPLICIT DIFFERENTIATION - DERIVATIVES OF INVERSE FUNCTIONS

We can use implicit differentiation to determine the derivative of any inverse function in terms of the original function. Specifically, we have the following.

Result 2.1. Suppose $f(x)$ is a one to one function and let $f^{-1}(x)$ be its inverse. Then the derivative of $f^{-1}(x)$ is

$$\frac{1}{f'(f^{-1}(x))}.$$

Proof. If $y = f^{-1}(x)$ we get $f(y) = x$. Differentiating both sides with respect to x , we get

$$f'(y)y' = 1$$

or

$$y' = \frac{1}{f'(y)}.$$

Substituting back in for y , we have

$$y' = \frac{1}{f'(f^{-1}(x))}.$$

□

We can use this to determine derivatives of the remaining functions we studied in the first few sections.

Result 2.2. If $f(x) = \ln(x)$, then

$$f'(x) = \frac{1}{x}.$$

Proof. If

$$y = \ln(x)$$

then $e^y = x$. Differentiating both sides with respect to x , we get $e^y y' = 1$, so $y' = \frac{1}{e^y}$. Substituting back in, we get

$$y' = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

□

Result 2.3. If $f(x) = \log_a(x)$, then

$$f'(x) = \frac{1}{\ln(a)x}$$

Proof. If

$$y = \log_a(x)$$

then $a^y = x$. Differentiating both sides with respect to x , we get $\ln(a)a^y y' = 1$, so $y' = \frac{1}{\ln(a)a^y}$. Substituting back in, we get

$$y' = \frac{1}{\ln(a)a^{\log_a(x)}} = \frac{1}{\ln(a)x}$$

□

Result 2.4. If $f(x) = \arcsin(x)$, then

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

Proof. If

$$y = \arcsin(x)$$

then $\sin(y) = x$. Differentiating both sides with respect to x , we get $\cos(y)y' = 1$, so $y' = \frac{1}{\cos(y)}$. Now observe that $x = \sin(y)$, so it follows that

$$\cos^2(y) + \sin^2(y) = 1$$

meaning

$$\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}$$

Thus we have

$$f'(x) = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

Note that we could have also solved this by substituting back in $\arcsin(x)$ for y and simplifying $\cos(\arcsin(x))$ using elementary trigonometry and triangles.

□

Result 2.5. If $f(x) = \arctan(x)$, then

$$f'(x) = \frac{1}{x^2 + 1}$$

Proof. If

$$y = \arctan(x)$$

then $\tan(y) = x$. Differentiating both sides with respect to x , we get $\sec^2(y)y' = 1$, so $y' = \frac{1}{\sec^2(y)}$. Now observe that $\sec^2(y) = \tan^2(y) + 1 = x^2 + 1$. Substituting in, we get

$$f'(x) = \frac{1}{x^2 + 1}.$$

□

Thus we can now differentiate (in principle) every type of function we discussed at the beginning of the course. We finish with a couple of examples.

Example 2.6. Differentiate the following:

(i)

$$f(x) = \arctan(x^2 + 1)$$

Applying the chain rule, we have

$$f'(x) = \frac{1}{1 + (x^2 + 1)^2} \cdot 2x = \frac{2x}{1 + (x^2 + 1)^2}.$$

(ii)

$$g(x) = \arcsin(\arctan(x))$$

Applying the chain rule, we have

$$f'(x) = \frac{1}{\sqrt{1 - (\arctan(x))^2}} \cdot \frac{1}{1 + x^2} = \frac{1}{(1 + x^2)\sqrt{1 - (\arctan(x))^2}}.$$