Section 3.6: Logarithmic Differentiation

In the previous section we learnt how to differentiate logarithms. In this section, we shall consider some elementary examples of derivatives of logarithms and then we shall see how we can use logarithms and their derivatives together with implicit differentiation to convert very difficult derivatives into much easier derivatives. The process is called logarithmic differentiation.

1. Derivatives of Logarithms

Recall that if

$$f(x) = \log_a(x)$$

then

$$f'(x) = \frac{1}{\ln(a)x}.$$

This means we can differentiate nearly any function which involves a logarithm. We illustrate with some examples.

Example 1.1. Differentiate the following.

(*i*) $f(x) = \ln(1 - e^{-x})$

Using the chain rule, we have

$$f'(x) = \frac{1}{1 - e^{-x}} \cdot e^{-x} = \frac{e^{-x}}{1 - e^{-x}}.$$

(*ii*)

$$g(x) = \frac{x}{1 + \log_2(x)}$$

Using the quotient rule, we have

$$g'(x) = \frac{(1 + \log_2(x)) - x\frac{1}{\ln(2)x}}{(1 + \log_2(x))^2} = \frac{(1 + \log_2(x)) - \frac{1}{\ln(2)}}{(1 + \log_2(x))^2}.$$

(*iii*)

$$k(x) = \ln(x^x(x+1)^2)$$

Before we differentiate, using log properties, we have

$$k(x) = \ln(x^x(x+1)^2) = x\ln(x) + 2\ln(x+1).$$

Now differentiating, we have

$$k'(x) = \ln(x) + x\frac{1}{x} + \frac{2}{x+1} = \ln(x) + 1 + \frac{2}{x+1}.$$

Observe that this last example was made much easier after we simplified using the log properties. We shall now consider how this method can be used to differentiate other complicated functions.

2. Logarithmic Differentiation

When we need to differentiate a function which involves lots of exponents and multiplication, we can use logarithms and implicit differentiation to make them easier. We illustrate with with an example.

Example 2.1. Differentiate

$$f(x) = \frac{(x+1)^2(x+2)^2(x+4)^2}{(x-1)(x+5)}.$$

This of course would be a very difficult derivative requiring the quotient rule, the product rule (multiple times) and the chain rule (multiple times). However, we shall not torture ourselves with differentiating this and instead we shall try to simplify it using logarithms. If we set

$$y = \frac{(x+1)^2(x+2)^2(x+4)^3}{(x-1)(x+5)}$$

then taking logarithms of both sides, we get

$$\ln(y) = \ln\left(\frac{(x+1)^2(x+2)^2(x+4)^2}{(x-1)(x+5)}\right).$$

Note that we can dramatically simplify the right hand side using the log properties. Specifically, we have

$$\ln(y) = 2\ln(x+1) + 2\ln(x+2) + 3\ln(x+4) - \ln(x-1) - \ln(x+5).$$

Now using implicit differentiation, we have

$$y'\frac{1}{y} = \frac{2}{x+1} + \frac{2}{x+2} + \frac{3}{x+4} - \frac{1}{x-1} - \frac{1}{x+5}.$$

Therefore, we have

$$y' = f'(x) = \frac{(x+1)^2(x+2)^2(x+4)^2}{(x-1)(x+5)}\left(\frac{2}{x+1}+\frac{2}{x+2}+\frac{3}{x+4}-\frac{1}{x-1}-\frac{1}{x+5}\right).$$

The method of using logarithms in this was is called **Logarithmic Differentiation**. For some functions, logarithmic differentiation can be used merely as a convenience to help with derivatives. However, for some functions, logarithmic differentiation is the only way to calculate a derivative. We illustrate the method of logarithmic differentiation through some examples.

Example 2.2. Differentiate the following functions.

(*i*)

$$f(x) = x^x$$

Notice that this is neither a power function or an exponential function, so neither of those rules can be used to differentiate $f(x)$. Instead, we have to apply logarithmic differentiation to differentiate this function. Specifically, if $y = x^x$, we have

$\ln(y) = \ln(x^x) = x\ln(x)$. Differentiating both sides with respect to x, we get

$$y'\frac{1}{y} = \ln(x) + x\frac{1}{x} = \ln(x) + 1.$$

Thus

$$f'(x) = y(\ln(x)+1) = x^x(\ln(x)+1).$$

(ii)

$$g(x) = \sqrt[4]{\frac{x^2+1}{x^2-1}}$$

This function we could evaluate using the chain rule and quotient rule. However, it will be much easier using logarithmic differentiation. Specifically, if

$$y = \sqrt[4]{\frac{x^2+1}{x^2-1}}$$

then

$$\ln(y) = \ln\left(\sqrt[4]{\frac{x^2+1}{x^2-1}}\right) = \frac{1}{4}(\ln(x^2+1) - \ln(x^2-1))$$

Differentiating both sides with respect to x, we get

$$y'\frac{1}{y} = \frac{1}{4}\left(\frac{2x}{x^2+1} - \frac{2x}{x^2-1}\right).$$

Thus

$$g'(x) = \frac{y}{4}\left(\frac{2x}{x^2+1} - \frac{2x}{x^2-1}\right) = \frac{1}{4}\sqrt[4]{\frac{x^2+1}{x^2-1}}\left(\frac{2x}{x^2+1} - \frac{2x}{x^2-1}\right).$$

(iii)

$$k(x) = (\ln(x))^x$$

Notice that this is neither a power function or an exponential function, so neither of those rules can be used to differentiate $k(x)$. Instead, we have to apply logarithmic differentiation to differentiate this function. Specifically, if $y = (\ln(x))^x$, we have $\ln(y) = \ln(\ln(x)^x) = x\ln(\ln(x))$. Differentiating both sides with respect to x, we get

$$y'\frac{1}{y} = \ln(\ln(x)) + x\frac{1}{\ln(x)}\frac{1}{x} = \ln(\ln(x)) + \frac{1}{\ln(x)}.$$

Thus

$$k'(x) = y\left(\ln(\ln(x)) + \frac{1}{\ln(x)}\right) = (\ln(x))^x\left(\ln(\ln(x)) + \frac{1}{\ln(x)}\right).$$

(*iv*)

$$r(x) = x^{e^x}$$

Notice that this is neither a power function or an exponential function, so neither of those rules can be used to differentiate $r(x)$. Instead, we have to apply logarithmic differentiation to differentiate this function. Specifically, if $y = x^{e^x}$, we have $\ln(y) = \ln(x^{e^x}) = e^x \ln(x)$. Differentiating both sides with respect to x, we get

$$y' \frac{1}{y} = e^x \ln(x) + \frac{e^x}{x}.$$

Thus

$$f'(x) = y\left(e^x \ln(x) + \frac{e^x}{x}\right) = x^{e^x}\left(e^x \ln(x) + \frac{e^x}{x}\right).$$