Section 3.9: Related Rates

Suppose that two quantities x and y are related by some equation. Then it follows that their derivatives must also be related by some equation (so we say they have **related rates**. In this section, we consider how, if we know the rate of change of one of these quantities, we can use implicit differentiation to determine the rate of change of the other.

1. Related Rates

Problems in related rates can become quite complicated. Therefore, we shall start with an easy example, then determine a general approach to the problem, and then finish with a number of more complicated examples.

Example 1.1. Suppose you are inflating a spherical balloon. Its volume V is measured in cm^3 and its radius r in cm.

(i) Find dV/dr when $r = 1$ and $r = 2$ and give a practical interpretation of your answers.

   We know the volume of a sphere is terms of its radius is given by

$$V = \frac{4}{3}\pi r^3.$$

   Therefore

$$\frac{dV}{dr} = 4\pi r^2.$$

   When $r = 1$, we have $dV/dr = 4\pi$ and when $r = 2$ we have $dV/dr = 16\pi$. This means that when $r = 1$, the volume is changing at a rate of $4\pi cm^3$ for each $1cm$ change in radius, and when $r = 2$, the volume is changing at a rate of $16\pi cm^3$ for each $1cm$ change in radius.

(ii) Suppose the balloon is being inflated so that $r(t) = 2t cm$ at any time t. How fast is the volume of the balloon changing with respect to time when $r = 1$ and $r = 2$?

   In this case, we are trying to find dV/dt. We know that

$$V = \frac{4}{3}\pi r^3$$

   and we also know that r is a function of t. Therefore, to get dV/dt, we can use implicit differentiation and differetiate both sides with respect to t. Specifically, we shall have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

(since we are differentiating r with respect to t). However, we also know $r = 2t$, so $dr/dt = 2$, and thus

$$\frac{dV}{dt} = 8\pi r^2.$$

Thus when $r = 1$, we have $dV/dt = 8\pi$ and when $r = 2$ we have $dV/dt = 32pi$.

(iii) If the balloon is being inflated at a constant rate of $5cm^3/s$, how fast is the radius of the balloon changing when $r = 1$ and $r = 2$?

In this case, we are trying to find dr/dt. We know that

$$V = \frac{4}{3}\pi r^3,$$

we know that r is a function of t and we also know that $dV/dt = 5$. Therefore, to get dr/dt, we can use implicit differentiation and differetiate both sides with respect to t getting

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

(since we are differentiating r with respect to t). Since $dV/dt = 5$, it follows that

$$5 = 4\pi r^2 \frac{dr}{dt}$$

and so

$$\frac{dr}{dt} = \frac{5}{4\pi r^2}.$$

Thus when $r = 1$, we have $dr/dt = \frac{5}{4\pi}$ and when $r = 2$ we have $dr/dt = \frac{5}{16\pi}$.

Notice that in the example, we wrote down the expression relating volume and radius, and when we wanted to determine the rate of change with respect to time, we differentiated with respect to t using implicit differentiation. This is one of the key steps in related rates. In general, the best way to approach a related rates formula is to use the following steps:

(i) Introduce all relevant variables and draw a picture to describe what is going on.

(ii) Write down the quantity you are trying to find - any related rates problem will be asking you to find some derivative. If you have written it down somewhere, it will give you a road map of where you are headed.

(iii) Write down what you know. Usually in a related rates problem you will be given certain rates. You should write them all down so you know what you are starting with.

(iv) Write down all relevant equations relating the variables. Use them to try to write down a single equation relating the variable for which you are trying to find the rate of change to the variables for which you know the rates of change.

(v) Differentiate this equation with respect to the relevant variable implicitly. Plug in the values you know and solve for the rate you are trying to find.

If you get into the habit of following these steps, then one of the hardest topics in calculus 1 will be much easier. We illustrate with some examples.

Example 1.2. A ladder $3m$ in length stands against a wall. The foot of the ladder moves outward at a speed of $0.1m/s$ when the foot of the ladder is $1m$ from the wall, how fast is the top of the ladder falling? What if it was $2m$ from the wall? Also, explain why your answers are negative.

We shall proceed through the recommended steps:

(i) Sketch a picture and introduce variables:
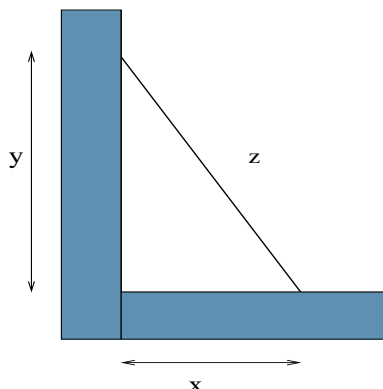
We need a variable x for the distance of the foot from the wall, a variable y for the height of the ladder, a variable z for the length of the ladder, and a variable t measuring time.

(ii) What we want: We are trying to find dy/dt, the rate of change of the height of the ladder with respect to time.

(iii) What we know: $dx/dt = 0.1$ since the foot of the ladder is moving out at that rate. We also know $z = 3$ because the ladder has length 3.

(iv) Equations relating the variables: We can use the Pythagoras to related the variables. Specifically, we have $x^2 + y^2 = z^2$. Since $z = 3$, it follows that $x^2 + y^2 = 9$. This equation relates the two variables x and y. Also note that we know dx/dt and we want dy/dt, so this equation should be sufficient.

(v) Implicitly differentiate and solve: Differentiating both sides with respect to t, we get

$$2y\frac{dy}{dt} + 2x\frac{dx}{dt} = 0$$

or

$$\frac{dx}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

When the foot of the ladder is $1m$ from the wall, we have $x = 1$ and so $y = \sqrt{9-1} = \sqrt{8}$ and thus

$$\frac{dx}{dt} = -\frac{1}{\sqrt{8}}\frac{dx}{dt}.$$

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When the foot of the ladder is $2m$ from the wall, we have $x = 2$ and so $y = \sqrt{9-4} = \sqrt{4}$ and thus

$$\frac{dx}{dt} = -\frac{2}{\sqrt{5}}\frac{dx}{dt}.$$

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Note that we are obtaining negative answers since the value of y is decreasing i.e. y is getting smaller.

Example 1.3. Two people are 50 feet apart. One of them (person A) starts walking north at a rate so that the angle shown in the diagram below is changing at a constant rate of 0.01 rad/min. Person B stays still watching person A. At what rate is distance between the two people changing when $\vartheta = 0.2$ radians?
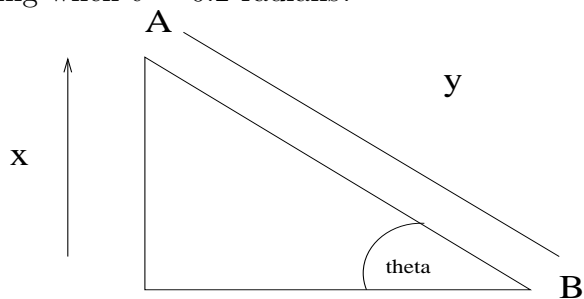
Let t represent time, let y be the distance from person A to person B at time t, and let x be the distance person A person A has traveled. Finally, let ϑ be the angle as given in the diagram. We want dy/dt and we know $d\vartheta/dt$, so we need to write down any equations relating these variables.

We have

$$x^2 + 50^2 = y^2$$

and

$$\cos(\vartheta) = \frac{50}{y}.$$

Since we want dy/dt and we know $d\vartheta/dt$, we differentiate the first equation with respect to ϑ:

$$-\sin(\vartheta)\frac{d\vartheta}{dt} = -\frac{50}{y^2}\frac{dy}{dt}.$$

We are are interested in dy/dt when $\vartheta = 0.2$ radians. Using this information, we have $y = 50/\cos(0.2) = 51.01$ and so

$$-\sin(0.2)\cdot 0.01 = -\frac{50}{(51.01)^2}\frac{dy}{dt}$$

giving

$$\frac{dy}{dt} = 0.1047\text{m/s}.$$

Note that the variable x was not used at all during this problem (sometimes not all variables will be used).

Example 1.4. A tank of water in the shape of a cone is leaking water at a constant rate of $2\text{ft}^3/\text{hr}$. The base radius of the tank is 5 ft and the height of the tank is 14 ft.

   (i) At what rate is the depth of the water in the tank changing when the depth of the water is 6ft?
   (ii) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?

First we sketch a picture. We have the following:
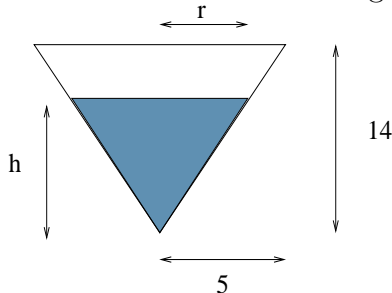
First we need a time variable t, and then, as suggested by the picture, we introduce the variables h and r to be the radius and the depth of the water at any given time t. We also know introduce the variable V for volume. For part (i) we are trying to find dh/dt and for part (ii) we are trying to find dr/dt. We are given $dV/dt = -2$. First, we note that the volume of the cone of water will be

$$V = \frac{1}{3}\pi r^2 h.$$

Next observe that r and h are related by similar triangles. Specifically, we have

$$\frac{r}{5} = \frac{h}{14},$$

so

$$r = \frac{5h}{14}$$

or
$$h = \frac{14r}{5}.$$

For (i), we are looking for dh/dt, so substituting in for r we get
$$V = \frac{1}{3}\pi\frac{25h^3}{196}.$$

Differentiating with respect to t, we get
$$\frac{dV}{dt} = \pi\frac{25h^2}{196}\frac{dh}{dt}$$
or
$$\frac{dh}{dt} = \frac{dV}{dt}\frac{196}{25\pi h^2}.$$

When $h = 6$, we have $dV/dt = -2$ and thus
$$\frac{dh}{dt} = -0.139 \text{ ft/hr.}$$

For (ii), we are looking for dr/dt, so substituting in for h we get
$$V = \frac{1}{3}\pi\frac{14r^3}{5}.$$

Differentiating with respect to t, we get
$$\frac{dV}{dt} = \pi\frac{14r^2}{5}\frac{dr}{dt}$$
or
$$\frac{dr}{dt} = \frac{dV}{dt}\frac{5}{14\pi r^2}.$$

When $h = 6$, we have $r = 30/14$ and $dV/dt = -2$ and thus
$$\frac{dr}{dt} = -0.0495 \text{ ft/hr.}$$