Section 4.1: Maximum and Minimum Values

In this chapter, we shall consider further applications of the derivative. The main application we shall consider is using derivatives to sketch accurate graphs of functions. Another important application we shall consider is optimization - the problem of optimizing or minimizing a quantity which depends upon another. An important aspect of both of these problems is being able to determine the minimum and maximum values of a function, so we shall start with this problem.

1. The Definitions of Minimum and Maximum Values

We start with the definition of a minimum or maximum value. There are four different definitions we shall consider:

Definition 1.1. (*i*) We say that $f(x)$ has an absolute maximum at $x = c$ if $f(c) \geqslant f(x)$ for all values in the domain of f.

(*ii*) We say that $f(x)$ has a local maximum at $x = c$ if $f(c) \geqslant f(x)$ for all x is some open interval containing c.

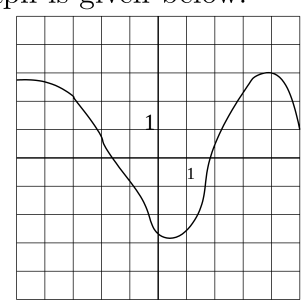
(*iii*) We say that $f(x)$ has an absolute minimum at $x = c$ if $f(c) \leqslant f(x)$ for all values in the domain of f.

(*iv*) We say that $f(x)$ has a local minimum at $x = c$ if $f(c) \leqslant f(x)$ for all x is some open interval containing c.

We consider some straight forward examples.

Example 1.2. Determine the global and local minimums and maximums of the following:

(*i*) $f(x)$ whose graph is given below.



Looking at the graph, this has a local minimum at $x = 1/2$ (which also looks like it may be a global minimum). There is a local maximum value at $x = 4$ which looks like it could also be a global maximum. There are no other mins or maxes.

(*ii*) $f(x) = x^2$ on the interval $[-1, 2]$

$f(x) = x^2$ has a local minimum value at $x = 0$. Since $x^2 \geqslant 0$ for all x, this must also be a global minimum. It doesn't have any local or global maxes on its whole domain, but since we are restricting, we need to see if there are any global or local maxes on $[-1, 2]$. We have $f(-1) = 1 \geqslant x^2$ for all x close to

$x = -1$ and greater than $x = -1$, so it is a local max. We have $f(2) = 4 \geqslant x^2$ for all x in the interval $[-1, 2]$, so it is a local and global max.

(iii) $f(x) = \frac{1}{x}$

$f(x) = 1/x$ has an asymptote at $x = 0$. Moreover, $\lim_{x \to 0^+} f(x) = \infty$ and $\lim_{x \to 0^-} f(x) = -\infty$. In particular, there can be no global mins or maxes. In addition to this, for $x > 0$, $f(x)$ is always decreasing and for $x < 0$ it is always increasing, so it cannot take a local max or min anywhere. Thus $f(x) = 1/x$ has no local or global maximums or minimums.

We can also consider different possibilities for when mins/maxes occur (or do not occur).

Example 1.3. (i) Can a function have infinitely many mins/maxes?

Yes - consider $f(x) = \sin(x)$.

(ii) Can a function have no mins/maxes?

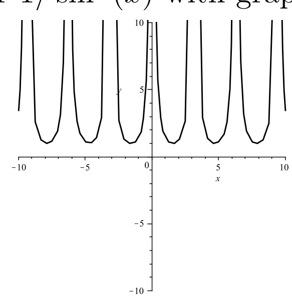
Yes - we saw $f(x) = 1/x$ has no mins or maxes.

(iii) Can a function one min and no maxes?

Yes - $f(x) = x^2$.

(iv) Can a function more than one min and no maxes?

Yes - consider $1/\sin^2(x)$ with graph illustrated below:



As seen in our examples and previous remarks, mins and maxes may or may not occur (either global or local). However, under certain additional conditions, we can guarantee that they will always exist. Specifically, we have the following result.

Result 1.4. (The Extreme Value Theorem) If $f(x)$ is continuous on $[a, b]$, then $f(x)$ takes a maximum and minimum value somewhere in te interval $[a, b]$.

The proof of this is beyond the scope of the course. However, we can illustrate why this fails when we drop some of the conditions.

Example 1.5. (i) Show that the condition "continuous" is a necessary condition for the extreme value theorem.

Consider the function $f(x) = 1/x$ on the interval $[-1, 1]$. On this interval, $f(x)$ has no maximum or minimum value. However, it does not contradict the extreme value theorem since it is not continuous on that interval.

(ii) Show that the interval must be closed in order for the extreme value theorem to hold.

Consider the function $f(x) = 1/x$ on the interval $(0, 1]$. On this interval, $f(x)$ has no maximum (since it grows without bound) even though it is continuous on this interval. However, it does not contradict the extreme value theorem since the interval $(0, 1]$ is not open.

2. Locating Minima and Maxima

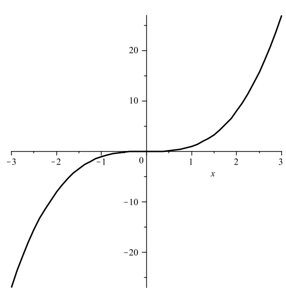
Now we have defined minimum and maximum values and seen a number of examples, we shall consider the most important topic - finding them. We shall see that in general, finding minimum and maximum values can be reduced to a fairly straight forward problem in Calculus and elementary algebra. The main result we shall use is the following:

Result 2.1. (Fermats Theorem) If $f(x)$ has a min or max at $x = c$ then $f'(c) = 0$ provided it exists.

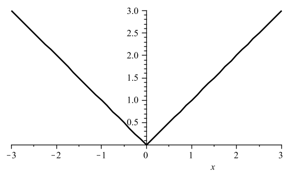
This is true simply because if $x = c$ is a maximum, then $f(x)$ increases as x moves toward c and decreases as it moves aways from c. In particular, the derivative $f'(x)$ goes from negative to positive, and hence must equal 0 at c. A similar argument holds when $x = c$ is a minimum.

Warnings

(i) Just because $f'(c) = 0$ does not mean $f(x)$ has a min or max at $x = c$. For example $f(x) = x^3$ has neither a min or max at $x = 0$, but $f'(0) = 0$.



(ii) $f(x)$ can have a min or max at $x = c$ without $f'(c) = 0$. Specifically, $f'(c) = 0$ only if the derivative exists there, so a function may have a min or max at a point where the derivative is undefined. For example $f(x) = |x|$ has a min at $x = 0$, but $f'(0) \neq 0$ since it does not exist.



This means one of two things must happen at a min or max. Either:

(i) $f'(x) = 0$
(ii) $f'(x)$ is undefined.

These observations motivate the following definition:

Definition 2.2. We say a point c is a critical point of $f(x)$ if $f'(c) = 0$ or $f'(c)$ is undefined.

Now suppose we are trying to find the global mins and maxes of a function $f(x)$ on an interval $[a, b]$. Then the min and max will either take place at one of the critical points (since it will be a local min or max too), or it will take place at one of the end points. This suggests the following method to locate the absolute mins and maxes of a function $f(x)$ on a closed interval $[a, b]$.

Result 2.3. To locate the absolute mins and maxes of a function $f(x)$ on a closed interval $[a, b]$ we do the following:

(i) Find all critical points and find the value of $f(x)$ at each.
(ii) Find the value of $f(x)$ at the two endpoints.
(iii) The largest value of the previous two steps is the max and the smallest is the min.

To illustrate this method, we finish with a couple of detailed examples.

Example 2.4. Locate the critical points of $f(x) = x^3 - 3x$ on $[-2, 2]$ and use you answer to find the absolute minimum and absolute maximum of $f(x)$ on $[-1, 3]$.

First, to locate the critical points we set

$$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1) = 0.$$

Since there are no points where the derivative is undefined, the critical points are at $x = \pm 1$. To find the global min and max, we evaluate the function at the critical points and at the endpoints: $f(-2) = -2$, $f(2) = 2$, $f(-1) = 2$, $f(1) = -2$. Thus the global max is 2 and the global min is -2 (note that it takes its global min and max at two different places).

Example 2.5. Locate the critical points of $f(x) = \frac{x}{(x-1)^2}$ on $[-2, 2]$ and use you answer to find the absolute minimum and absolute maximum of $f(x)$ on $[-2, 2]$.

Using the chain rule, we have

$$f'(x) = \frac{(x-1)^2 - 2x(x-1)}{(x-1)^4} = \frac{x^2 - 2x + 1 - 2x^2 + 2x}{(x-1)^4} = \frac{-x^2+1}{(x-1)^4}$$

$$= -\frac{(x-1)(x+1)}{(x-1)^4} = -\frac{(x+1)}{(x-1)^3}$$

for $x \neq 1$. Thus the critical points are at $x = 1$ (where the derivative is undefined) and $x = -1$. Evaluating, we have $\lim_{x\to 1^+} f(x) = \infty$ and $\lim_{x\to 1^-} f(x) = \infty$, so $f(x)$ has no maximum. Also, $f(-1) = -1/4$, and at the endpoints we have $f(-2) = -2/9$ and $f(2) = 2$. Thus $-1/4$ must be the global minimum which takes place at $x = -1$.

Example 2.6. Find the best viewing window for $f(x) = x^3 - 6x^2 + 9x + 5$ on $[0, 5]$.

To find an appropriate viewing window, we need to determine the largest and smallest y values this function takes on the interval $[0, 5]$. Therefore, this is simply a global min/max problem. Finding the critical points, we have

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3) = 0$$

when $x = 3$ and $x = 4$. Since the derivative is defined everywhere, the only critical points are $x = 3$ and 4. Evaluating at the critical points and endpoints, we get $f(0) = 5$, $f(1) = 9$, $f(3) = 5$ and $f(5) = 25$. Thus the global max is 25 and the global min is 25 and the global min is 5 (which is attained at two separate places).