Section 4.2: The Mean Value Theorem

Before we continue with the problem of describing graphs using calculus, we shall briefly pause to examine some interesting applications of the derivative. In previous sections, we examined the intermediate value theorem - a result which guaranteed that a function had to take certain values at certain points. In this section, we consider similar ideas for the derivative.

1. Rolles Theorem

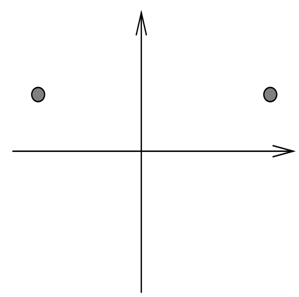
Before we consider the mean value theorem, we consider a related result.

Result 1.1. (Rolle's Theorem) Suppose that $f(x)$ satisfies the following three properties:

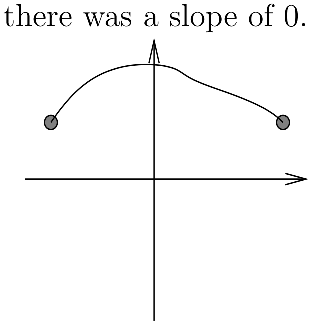
(i) f is continuous on the closed interval $[a, b]$,
(ii) f is differentiable on the open interval (a, b),
(iii) $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$, or equivalently, $f(x)$ has a horizontal tangent line for some c in (a, b).

Naively, this is completely obvious. If we draw any two points with the same y-value in the plane as follows:



then if we try to connect these points with the graph of a continuous and differentiable function, it will have to have a horizontal tangent line somewhere. Specifically, if $f(x)$ is not constant, then it will either have to increase at some point or decrease at some point. Consequently, to get back to the same y-value, it will have to decrease or increase later, and thus somewhere in between it must have switched from increasing to decreasing meaning there was a slope of 0.



We illustrate with some examples.

Example 1.2. Show that $f(x) = \sin(2\pi x)$ satisfies Rolle's Theorem on $[-1, 1]$ and then find all numbers which satisfy the conclusion.

Clearly $f(x)$ is differentiable and continuous on the interval $[-1, 1]$. Plugging in the endpoints, we have $f(-1) = 0 = f(1)$. This means Rolle's Theorem applies, so there must be some c in $(-1, 1)$ such that $f'(c) = 0$. Specifically, we have $f'(x) = 2\pi\cos(2\pi x)$, and this is equal to 0 when $x = -3/4, -1/4, 1/4$ and $3/4$ (so there are four values of c such that $f'(c) = 0$ on $(-1, 1)$).

Example 1.3. Explain using Rolle's Theorem why $f(x) = x + e^x$ has only one zero.

We know

$$\lim_{x\to\infty} x + e^x = \infty$$

and

$$\lim_{x\to-\infty} x + e^x = -\infty,$$

so since $f(x)$ is continuous, it must pass through the x-axis at some point. In particular, there must be some zero of $f(x)$ at some point. Now suppose there are two zeros - one at $x = a$ and one at $x = b$. Then on the interval $[a, b]$, we have $f(a) = 0 = f(b)$, and $f(x)$ is differentiable and continuous on this interval. In particular, Rolle's Theorem applies, so there must be c in $[a, b]$ such that $f'(c) = 0$. However, $f'(x) = 1 + e^x > 0$, so Rolle's Theorem can't apply and thus $f(x)$ cannot have a second zero.

Example 1.4. Explain why the condition "$f(x)$ is differentiable" is necessary in Rolle's Theorem.

Consider the function $f(x) = |x|$ on the interval $[-1, 1]$. Clearly we have $f(-1) = f(1)$, and so it seems that Rolle's Theorem should apply. However, $f(x)$ has no horizontal tangent lines, so Rolle's Theorem obviously does not apply. This seems like it contradicts Rolle's Theorem, but it does not since $f(x)$ needs to be differentiable for Rolle's Theorem to apply (which it is not at $x = 0$).

2. The Mean Value Theorem

Rolle's Theorem is actually a special case of a much more general result about the values a derivative can take.

Result 2.1. (The Mean Value Theorem or MVT) Suppose that $f(x)$ satisfies the following two properties:

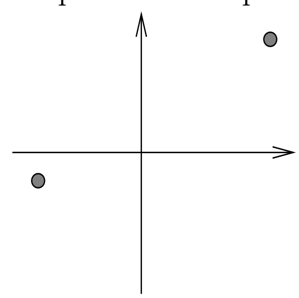
(i) f is continuous on the closed interval $[a, b]$,
(ii) f is differentiable on the open interval (a, b),

Then there is a number c in (a, b) such that

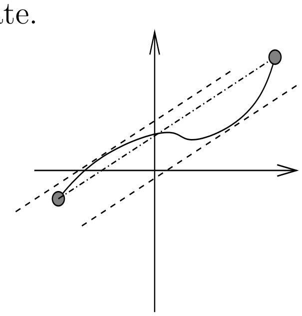
$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

or equivalently, $f(x)$ has a tangent line whose slope is equal to the slope of the line connecting the endpoints of $f(x)$ on (a, b) (or the average slope over the interval).

As with the Rolle's theorem, naively, it is completely obvious why this is true. If we draw any two points in the plane as follows:



then if we sketch the line between the two endpoints, there will be some tangent line between the points whose slope is equal to the slope of that line. We illustrate.



We illustrate with some examples.

Example 2.2. Find the values which satisfy the mean value theorem for $f(x) = e^{-2x}$ on the interval $[0, 3]$.

First we note that $f(x)$ is differentiable and continuous on $[0, 3]$, so MVT applies. First, the average slope is

$$\frac{f(3) - f(0)}{3 - 0} = \frac{e^{-6} - 1}{3}.$$

Next the derivative is $f'(x) = -2e^{-2x}$. MVT says that there is some c in $[0, 3]$ such that

$$f'(c) = \frac{e^{-6} - 1}{3},$$

so we can find c by setting these equal to each other and solving. Specifically, we have

$$-2e^{-2x} = \frac{e^{-6} - 1}{3},$$

$$e^{-2x} = \frac{1-e^{-6}}{6}$$

$$-2x = \ln\left(\frac{1-e^{-6}}{6}\right)$$

giving

$$x = -\frac{1}{2}\ln\left(\frac{1-e^{-6}}{6}\right) \simeq 0.897.$$

Example 2.3. Suppose f is continuous and differentiable for all x, $f(3) = 5$ and $-2 \geqslant f'(x) \leqslant 3$ for all x. What is the smallest and largest values $f(5)$ could be?

Using MVT, we know

$$\frac{f(5)-f(3)}{5-3} = f'(c)$$

for some c in $[3,5]$. Since $-2 \leqslant f'(x) \geqslant 5$, it follows that

$$\frac{f(5)-f(3)}{5-3} = f'(c) \leqslant 3$$

and

$$\frac{f(5)-f(3)}{5-3} = f'(c) \geqslant -2.$$

Thus

$$f(5) - f(3) \leqslant 6$$

or

$$f(5) \leqslant 11$$

and

$$f(5) - f(3) \geqslant -4$$

or

$$f(5) \geqslant 1.$$

Thus we have

$$1 \leqslant f(5) \leqslant 11.$$

3. Applications

There are two very important consequences of the mean value theorem. They will be important to us at the end of chapter 4 and for all of chapter 5 (and will be relevant to those taking Calculus 2).

Result 3.1. If $f'(x) = 0$ for all x in (c,d), then $f(x) = k$, a constant, on (c,d).

Proof. We just need to show that for a fixed a in (c, d), $f(a) = f(b)$ for any other number b in (c, d). To see this, we apply MVT to the interval $[a, b]$. Since f is differentiable and continuous on (c, d), it will be on $[a, b]$ too. Thus there is a number n in (a, b) such that

$$f'(n) = \frac{f(b) - f(a)}{b - a}.$$

Since the derivative is 0 everywhere, it follows that $f'(n) = 0$, so

$$\frac{f(b) - f(a)}{b - a} = 0$$

or $f(a) = f(b)$. Thus all values of $f(x)$ are equal for x in (c, d) and thus $f(x) = k$ for some constant k.

□

A consequence of this is the following.

Result 3.2. If $f'(x) = g'(x)$ on $[a, b]$, then $f(x) = g(x) + k$ for some constant k.

Proof. We simply apply the last result to the function $f - g$ to get the result.

□