Sections 4.3, 4.5 & 4.6: Graphing

In this section, we shall see how facts about $f'(x)$ and $f''(x)$ can be used to supply useful information about the graph of $f(x)$. Since there are three sections devoted to this topic, we shall combine all three into one.

1. How Derivatives Affect the Shape of a Graph

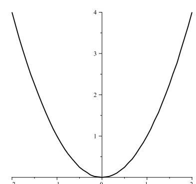
Before we start to examine the shape of the graph of a function $f(x)$, we recall a some definitions about the shape of a graph from algebra.

Definition 1.1. (*i*) A function $f(x)$ is increasing over an interval (a, b) if $f(x_1) > f(x_2)$ for any x_1, x_2 in (a, b) with $x_1 > x_2$.

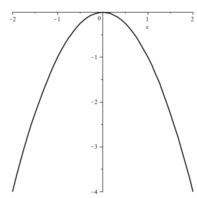
(*ii*) A function $f(x)$ is decreasing over an interval (a, b) if $f(x_1) > f(x_2)$ for any x_1, x_2 in (a, b) with $x_1 < x_2$.

(*iii*) A function $f(x)$ is concave up over an interval (a, b) if $f'(x_1) > f'(x_2)$ for any x_1, x_2 in (a, b) with $x_1 > x_2$ (so the slope of $f(x)$ is increasing over the interval (a, b)).

(*iv*) A function $f(x)$ is concave down over an interval (a, b) if $f'(x_1) > f'(x_2)$ for any x_1, x_2 in (a, b) with $x_1 < x_2$ (so the slope of $f(x)$ is decreasing over the interval (a, b)).

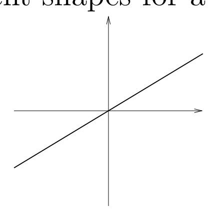


Concave up

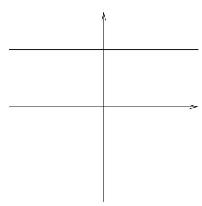


Concave Down

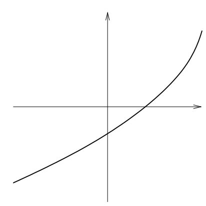
From these definitions, we can conclude that there are really only six different shapes for a graph:



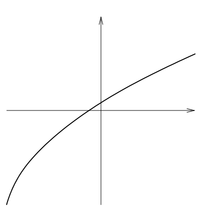
Linear



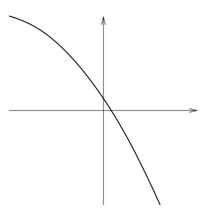
Constant



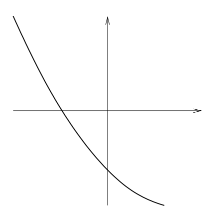
Increasing, Concave Up



Increasing, Concave down



Decreasing, Concave down

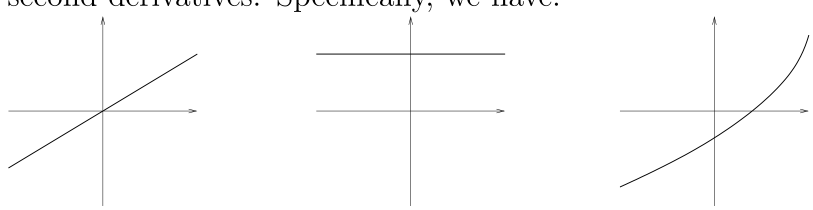


Decreasing, Concave Up

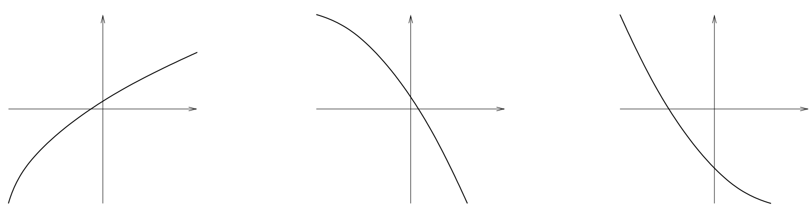
Since the shape of a graph is determined by concavity and whether it is increasing or decreasing, it follows that we can use derivatives to help find the shape of a graph. Specifically, we have the following:

(i) $f(x)$ is increasing on (a, b) if and only if $f'(x) > 0$ on (a, b).
(ii) $f(x)$ is decreasing on (a, b) if and only if $f'(x) < 0$ on (a, b).
(iii) $f(x)$ is concave up on (a, b) if and only if $f''(x) > 0$ on (a, b).
(iv) $f(x)$ is concave down on (a, b) if and only if $f''(x) > 0$ on (a, b).

Thus we can determine the shape of a graph by simply looking at first and second derivatives. Specifically, we have:



$f'(x) = C,\ f''(x) = 0$ $\quad$ $f'(x) = 0$ $\quad$ $f'(x) > 0,\ f''(x) > 0$



$f'(x) > 0,\ f''(x) < 0$ $\quad$ $f'(x) < 0,\ f''(x) < 0$ $\quad$ $f'(x) < 0,\ f''(x) < 0$

Before we proceed we need one more definition.

Definition 1.2. If $f(x)$ changes concavity at $x = a$ we call the point $(a, f(a))$ an inflection point of $f(x)$.

2. Using Derivatives to Determine the Basic Shape of a Graph

From our observations in the previous section, the shape of the graph of $f(x)$ can be determined by looking at the first and second derivatives of $f(x)$. This suggests the following method to determine the shape of $f(x)$ over an interval:

(i) Calculate $f'(x)$ and determine all critical points (these are the points where $f(x)$ can change from increasing to decreasing).
(ii) Plot all the critical points of $f(x)$ on an axis.
(iii) Subdivide the interval up into smaller intervals whose end-points are the critical points of $f(x)$. Test the sign of $f'(x)$ in each of these subintervals:
   (a) If $f'(x) > 0$ in a subinterval, $f(x)$ is increasing in that subinterval.
   (b) If $f'(x) < 0$ in a subinterval, $f(x)$ is decreasing in that subinterval.

(c) If $f'(x) = 0$ in a subinterval, $f(x)$ is constant in that subinterval.

(*iv*) Calculate $f''(x)$ and determine all points where $f''(x)$ is undefined or equal to zero (these are the points where $f(x)$ could have an inflection point i.e. change concavity).

(*v*) Plot all the points found in (*iv*) on the same axis.

(*vi*) Subdivide the interval up into smaller intervals whose endpoints are the points (*iv*). Test the sign of $f''(x)$ in each of these subintervals:

(a) If $f''(x) > 0$ in a subinterval, $f(x)$ is concave up in that subinterval.

(b) If $f''(x) < 0$ in a subinterval, $f(x)$ is concave down in that subinterval.

(c) If $f''(x) = 0$ in a subinterval, $f(x)$ is constant or linear in that subinterval.

(*vii*) Since we have all the points where concavity can change and $f(x)$ can change from increasing to decreasing (or vice versa), the shape of $f(x)$ anywhere between these points must be one of the six shapes given above. Moreover, from our work above, we can work out exactly which one. Thus we can now join all the points together with the correct shape graph over each of the subintervals.

We illustrate with a couple of examples.

Example 2.1. Sketch an accurate graph of $f(x) = x^3 - 3x + 1$ over its entire domain.

First we calculate the first and second derivatives:

$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1),$$

$$f''(x) = 6x.$$

Since $f'(x)$ and $f''(x)$ are defined everywhere, we only need to determine when they are zero.
Starting with $f'(x)$, we have $f'(x) = 0$ when $x = \pm 1$. Breaking into intervals, we need to test $f'(x)$ on the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$. We have

Interval	Test Point	Inc/Dec
$(-\infty, -1)$	$f'(-2) > 0$	Inc
$(-1, 1)$	$f'(0) < 0$	Dec
$(1, \infty)$	$f'(2) > 0$	Inc

Next we plot these points on an axis. We have $f(-1) = 3$ and $f(1) = -1$.
Next we consider $f''(x)$. We have $f''(x) = 6x = 0$ when $x = 0$. Breaking into intervals, we need to test $f''(x)$ on the intervals $(-\infty, 0)$ and

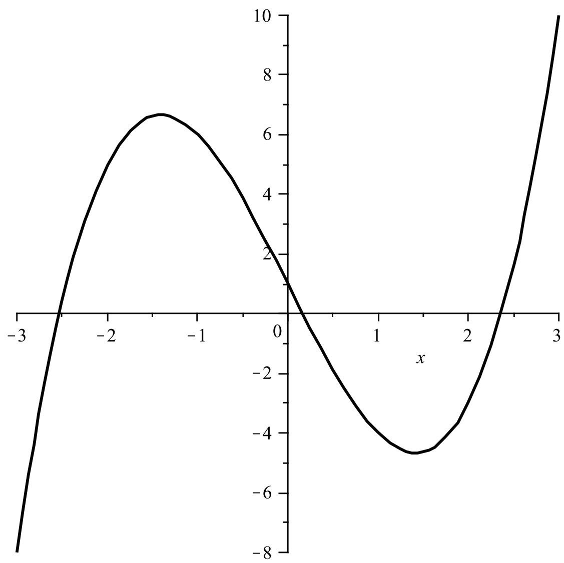
$(0, \infty)$. We have

Interval	Test Point	CU/CD
$(-\infty, 0)$	$f''(-1) < 0$	CD
$(0, \infty)$	$f''(1) < 0$	CU

This means there is an inflection point at $x = 0$. Next we plot this point on an axis. We have $f(0) = 1$.
Finally, for convenience, we shall examine the end behavior. Since $f(x)$ is a third degree polynomial with positive leading term, we have

$$\lim_{x\to\infty} f(x) = \infty \text{ and } \lim_{x\to-\infty} f(x) = -\infty.$$

Putting all this together, we get the following rough sketch of $f(x)$:



Example 2.2. Sketch the graph of $f(x) = e^x \sin(x)$ on the interval $[0, 2\pi]$.

First we calculate the first and second derivatives:

$$f'(x) = e^x \sin(x) + e^x \cos(x) = e^x(\sin(x) + \cos(x)),$$

$$f''(x) = e^x \sin(x) + e^x \cos(x) + e^x \cos(x) - e^x \sin(x) = 2e^x \cos(x).$$

Since $f'(x)$ and $f''(x)$ are defined everywhere, we only need to determine when they are zero.
Starting with $f'(x)$, we have

$$f'(x) = e^x(\sin(x) + \cos(x)) = 0$$

when

$$\sin(x) = -\cos(x)$$

or when $x = 3\pi/4$ and $x = 7\pi/4$. Breaking into intervals, we need to test $f'(x)$ on the intervals $(0, 3\pi/4)$, $(3\pi/4, 7\pi/4)$ and $(7\pi/4, 2\pi)$. We have

Interval	Test Point	Inc/Dec
$(0, 3\pi/4)$	$f'(\pi/6) > 0$	Inc
$(3\pi/4, 7\pi/4)$	$f'(\pi) < 0$	Dec
$(7\pi/4, 2\pi)$	$f'(11\pi/6) > 0$	Inc

Next we plot these points on an axis. We have $f(3\pi/4) = 1.55$ and $f(7\pi/4) = -172.64$.
Next we consider $f''(x)$. We have

$$f''(x) = 2e^x \cos(x) = 0$$

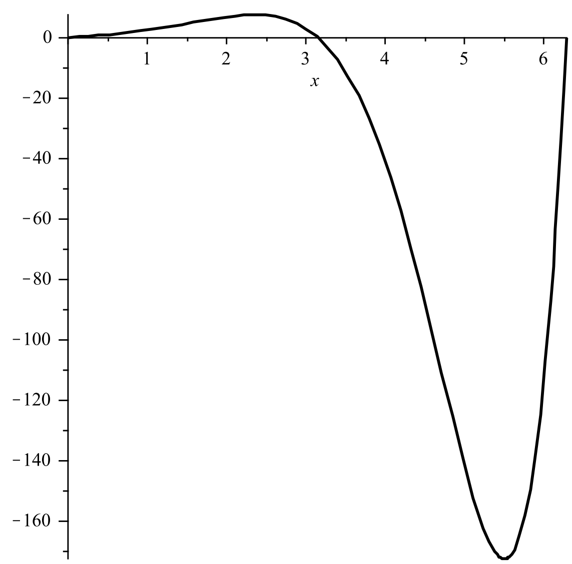
when

$$\cos(x) = 0$$

or when $x = \pi/2$ and $x = 3\pi/2$. Breaking into intervals, we need to test $f''(x)$ on the intervals $(0, \pi/2)$, $(\pi/2, 3\pi/2)$ and $(3\pi/2, 2\pi)$. We have

Interval	Test Point	CU/CD
$(0, \pi/2)$	$f''(\pi/4) > 0$	CU
$(\pi/2, 3\pi/2)$	$f''(\pi) < 0$	CD
$(3\pi/2, 2\pi)$	$f''(7\pi/4) > 0$	CU

This means that there are inflection points at both $x = \pi/2$ and $x = 3\pi/2$. Next we plot these points on an axis. We have $f(\pi/2) = 4.81$ and $f(3\pi/2) = -111.31$.
Finally, for convenience, we shall plot the endpoints. We have $f(0) = f(2\pi) = 0$. Putting all this together, we get the following rough sketch of $f(x)$:

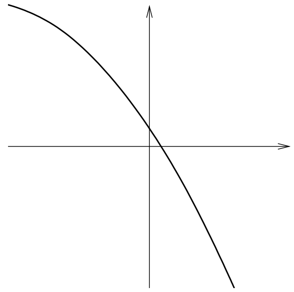


We look at a couple of other related problems.

Example 2.3. Answer the following true/false:

(i) There is a function $f(x)$ with $f(x) > 0$, $f'(x) < 0$ and $f''(x) < 0$ for all x.

False: Concave down decreasing would look like



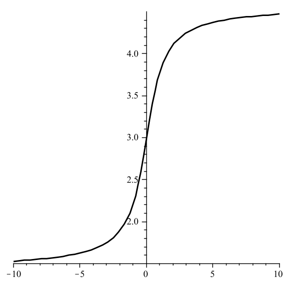
and thus would have to cross the x-axis at some point meaning $f(x) < 0$ at some point.

(ii) There is a function $f(x)$ with $f(x) > 0$, $f'(x) < 0$ and $f''(x) > 0$ for all x.

True: The function $f(x) = e^{-x}$ is an example.

(iii) If $f'(x) > 0$ and $f(x) > 0$ for all x then $\lim_{x\to\infty} f(x) = \infty$.

False: We could concave down increasing as $x \to \infty$ and concave up increasing as $x \to -\infty$. Such a graph would look like



which may have a horizontal asymptote (and thus $\lim_{x\to\infty} f(x) \neq \infty$).

3. Finding Local Minimum and Maximum Values

In the previous section we discussed how we can use derivatives to find the basic shape of a graph. In this section we shall discuss how derivatives can be used to find local minimum and maximum values which will help when trying to sketch the graph of a function. First we make some observations:

(i) Suppose $f(x)$ has a maximum value at $x = a$. Then the following will be true:
  (a) $f'(x)$ will either be zero or undefined
  (b) for $x < a$ but close to a we have $f'(x) > 0$ and for $x > a$ but close to a we have $f'(x) < 0$ i.e. $f(x)$ increases

towards the maximum value and decreases away from the maximum value

(c) at a maximum value, the concavity must either be concave up, $f''(x) > 0$, or zero

(*ii*) Suppose $f(x)$ has a minimum value at $x = a$. Then the following will be true:

(a) $f'(x)$ will either be zero or undefined

(b) for $x < a$ but close to a we have $f'(x) < 0$ and for $x > a$ but close to a we have $f'(x) > 0$ i.e. $f(x)$ decreases towards the maximum value and increases away from the maximum value

(c) at a maximum value, the concavity must either be concave down, $f''(x) > 0$, or zero

This suggests the following two methods to determine whether a critical point is a minimum or maximum value:

Result 3.1. (The first derivative test) Suppose c is a critical point of $f(x)$.

(*i*) If $f'(x)$ changes from negative to positive at $x = c$, then $x = c$ is a local miniumum.

(*ii*) If $f'(x)$ changes from positive to negative at $x = c$, then $x = c$ is a local maximum.

(*iii*) If $f'(x)$ does not change sign at $x = c$, then $x = c$ is neither a local minimum or maximum.

Result 3.2. (The second derivative test) Suppose c is a critical point of $f(x)$.

(*i*) If $f''(x) < 0$ then $x = c$ is a local maximum.

(*ii*) If $f''(x) > 0$ then $x = c$ is a local minimum.

(*iii*) If $f''(x) = 0$ then the test is inconclusive and we need to use the first derivative test to determine the nature of c.

Of the two tests, the second derivative test is usually the easiest, but it is also sometimes inconclusive. The first derivative test is never inconclusive, but is usually more time consuming to apply. We look at some examples to illustrate how to use these tests.

Example 3.3. Classify the critical points of the following functions:

(*i*) $g(x) = xe^{-x}$

We have

$$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x).$$

Since $f'(x)$ is never undefined, the only critical point is at $x = 1$. To determine the nature of this critical point, we shall apply the second derivative test. We have

$$f''(x) = -e^{-x} - e^{-x}(1 - x) = -e^{-x}(2 - x).$$

Then $f''(1) = -e^{-1} < 0$, and hence there is a local maximum at $x = 1$.

(ii) $f(x) = x + 1/x$

We have

$$f'(x) = 1 - \frac{1}{x^2}.$$

Since $f'(x)$ only undefined at $x = 0$ and $f(x)$ is also undefined at that point, a min or max cannot occur at that point. Therefore, the only possible critical points where $f(x)$ could have a max or min are when $f'(x) = 0$ or $x = \pm 1$. To determine the nature of these critical points, we shall apply the second derivative test. We have

$$f''(x) = \frac{1}{x^3}.$$

Then $f''(1) = 1 > 0$ and $f''(-1) = -1 < 0$, and hence there is a local maximum at $x = -1$ and a local minimum at $x = 1$.

(iii) $f(x) = 3x^5 - 5x^3$

We have

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x-1)(x+1).$$

Since $f'(x)$ is never undefined, the critical points are at $x = 0, \pm 1$. To determine the nature of this critical point, we shall apply the second derivative test. We have

$$f''(x) = 60x^3 - 30x.$$

Then $f''(1) = 30 > 0$, $f''(-1) = -30$ and $f''(0) = 0$, hence there is a local maximum at $x = -1$, a local minimum at $x = 1$ and the test is inconclusive at $x = 0$. Since the test is inconclusive at $x = 0$, we need to apply the first derivative test. Notice that for $x < 0$ but close to 0, we have $f'(x) = 15x^2(x-1)(x+1) < 0$ and for $x > 0$ but close to 0, we have $f'(x) = 15x^2(x-1)(x+1) < 0$. In particular, the sign does not change at $x = 0$ and hence there is neither a minimum or maximum at $x = 0$.

(iv) $f(x) = x \ln(x)$ for $x > 0$

We have

$$f'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1.$$

Since $f'(x)$ is only undefined when $f(x)$ is undefined, the only critical points occur when $f'(x) = 0$. Solving for x, we get $\ln(x) = -1$ or $x = e^{-1}$. To determine the nature of this critical point, we shall apply the second derivative test. We have

$$f''(x) = \frac{1}{x}.$$

Then $f'(e^{-1}) = \frac{1}{e^{-1}} > 0$, and hence there is a local minimum at $x = e^{-1}$.

4. Guidelines for Sketching Graphs

We have already seen how to use derivatives to get a basic sketch of a graph. In order to make it more accurate, we can plot additional points and utilize all our previous work on properties of functions. In general, to sketch an accurate graph of a function $f(x)$, we should use the following steps (note that the first 4 steps are precalculus and the last two steps are what we discussed at the beginning of this section):

- (i) Find the domain of $f(x)$
- (ii) Determine all intercepts and plot them
- (iii) Determine any symmetry (odd, even, periodic)
- (iv) Find any asymptotes (vertical and horizontal) and more generally, the end behavior
- (v) Find the intervals of increasing and decreasing and plot the critical points and minimums and maximums
- (vi) Find the intervals of concave up and concave down and plot the points where $f''(x)$ is zero or undefined (the inflection points)

We illustrate with three more examples.

Example 4.1. Sketch an accurate graph of $f(x) = e^x - 10x$

We shall proceed with each of the steps outlined above.

- (i) The domain of $f(x)$ is all real numbers.
- (ii) The x intercepts are when $f(x) = 0$ or $e^x - 10x = 0$. This is a transcendental function with a linear function so it cannot be solve by traditional means, so we use our calculator and we get two solutions: $(0.11, 0)$ and $(3.38, 0)$. The y-intercept is $(0, 1)$. We next plot these points on an axis.
- (iii) The function has no symmetry.
- (iv) There are no vertical asymptotes. Also, we have

$$\lim_{x\to\infty} f(x) = \infty \text{ and } \lim_{x\to-\infty} f(x) = \infty.$$

- (v) We have

$$f'(x) = e^x - 10.$$

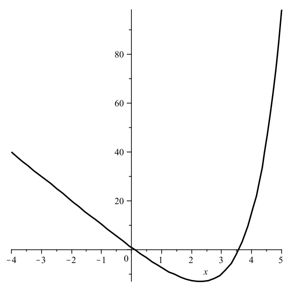
Since this is defined for all x, the only critical points occur when $f'(x) = 0$ or when $x = \ln(10))$. Therefore we need to test on the intervals $(-\infty, \ln(10)$ and $(\ln(10), \infty)$. We have

Interval	Test Point	Inc/Dec
$(-\infty, \ln(10))$	$f'(0) < 0$	Dec
$(\ln(10), \infty)$	$f'(3) > 0$	Inc

This means that $f(x)$ has a minimum value at $x = \ln(10)$ with value $f(\ln(10)) = 10 - 10\ln(10)$.

(*vi*) The second derivative is $f''(x) = e^x$ which is never zero or undefined. This means there is never a change in concavity. Since $f''(x) = e^x > 0$ for all x, it follows that $f(x)$ is concave everywhere.

Putting all this information together, we can now sketch a graph of $f(x)$:



Example 4.2. Sketch an accurate graph of $f(x) = xe^{-x^2}$

(*i*) The domain of $f(x)$ is all real numbers.

(*ii*) The x intercepts are when $f(x) = 0$ or $xe^{-x^2} = 0$. This is only zero when $x = 0$. Thus there is only one x-intercept which is also the intercept - $(0, 0)$.

(*iii*) The function is odd since $f(x) = xe^{-x^2} = -f(-x)$ i.e. it has symmetry about the origin. This means we only need to concentrate on the graph for $x \geqslant 0$.

(*iv*) There are no vertical asymptotes. Also, we have

$$\lim_{x\to\infty} f(x) = 0 \text{ and } \lim_{x\to-\infty} f(x) = 0$$

using L'Hoptials rule (since $f(x) = x/e^{x^2}$).

(*v*) We have

$$f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1 - 2x^2).$$

Since this is defined for all x, the only critical points occur when $f'(x) = 0$ or $x = \pm 1/\sqrt{(2)}$. Therefore we need to test on the intervals $(-\infty, -1/\sqrt{(2)})$, $(-1/\sqrt{(2)}, 1/\sqrt{(2)})$ and $(1/\sqrt{(2)}, \infty)$. We have

Interval	Test Point	Inc/Dec
$(-\infty, -1/\sqrt{(2)})$	$f'(-10) < 0$	Dec
$(-1/\sqrt{(2)}, 1/\sqrt{(2)})$	$f'(0) > 0$	Inc
$(1/\sqrt{(2)}, \infty)$	$f'(10) < 0$	Dec

This means that $f(x)$ has a minimum value at $x = -1/\sqrt{(2)}$ and a maximum at $x = 1/\sqrt{(2)}$. We can plot these points on the graph

(vi) The second derivative is

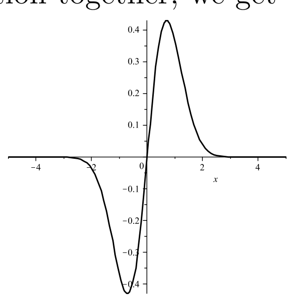
$$f''(x) = -4xe^{-x^2} - 2xe^{-x^2}(1 - 2x^2) = 2xe^{-x^2}(2x^2 - 3)$$

which is never undefined. This means the changes in concavity can only occur when $f''(x) = 0$, so when $x = 0$ or $x = \pm\sqrt{3}/\sqrt{2}$. Therefore we need to test on the intervals $(-\infty, -\sqrt{3}/\sqrt{(2)})$, $(-\sqrt{3}/\sqrt{(2)}, 0)$, $(0, \sqrt{3}/\sqrt{2})$, and $(\sqrt{3}/\sqrt{(2)}, \infty)$. We have

Interval	Test Point	CU/CD
$(-\infty, -\sqrt{3}/\sqrt{(2)})$	$f''(-10) < 0$	CD
$(-\sqrt{3}/\sqrt{(2)}, 0)$	$f''(-1) > 0$	CU
$(0, \sqrt{3}/\sqrt{2})$	$f''(1) < 0$	CD
$(\sqrt{3}/\sqrt{(2)}, \infty)$	$f''(10) > 0$	CU

This means there are inflection points at $x = 0, \pm\sqrt{3}/\sqrt{2}$.

Putting all this information together, we get the following graph:



Note that we could have used the symmetry and actually performed only half of the work.

Example 4.3. Sketch an accurate graph of

$$f(x) = \frac{x^2}{x^2 + 1}$$

(i) The domain of $f(x)$ is all real numbers.

(ii) The x intercepts are when $f(x) = 0$ and this occurs when $x = 0$. Since this also coincides with the y-intercept, it follows that $(0, 0)$ is the only intercept.

(*iii*) The function is even so is symmetric across the y-axis. In particular, we only need to concentrate on $x \geqslant 0$.

(*iv*) There are no vertical asymptotes. Also, we have

$$\lim_{x\to\infty} f(x) = 1 \text{ and } \lim_{x\to-\infty} f(x) = 1$$

using the leading term property.

(*v*) We have

$$f'(x) = \frac{2x(x^2+1) - 2x^3}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}.$$

Since this is defined for all x, the only critical points occur when $f'(x) = 0$ or when $x = 0$. Since we only need consider $x \geqslant 0$, we shall just test the intervals $(0, \infty)$. We have $f'(1) > 0$ and so $f(x)$ is increasing on the interval $(0, \infty)$.

(*vi*) The second derivative is

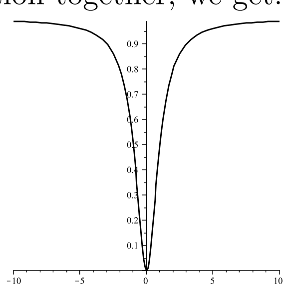
$$f''(x) = -\frac{2(3x^2-1)}{(x^2+1)^3}$$

which is never undefined and is equal to zero for $x = \pm 1/\sqrt{3}$. Since we are concentrating on $x \geqslant 0$, we need to test on the intervals $(0, 1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$. We have

Interval	Test Point	CU/CD
$(0, 1/\sqrt{3})$	$f''(1/2) > 0$	CU
$(1/\sqrt{3}, \infty)$	$f''(1) < 0$	CD

This means there are inflection points at $x = \pm 1/\sqrt{3}$ (by the symmetry).

Putting all this information together, we get:



5. Using a Calculator to aid in the Graphing Process

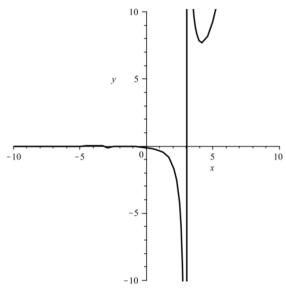
Sometimes a graphing calculator can be very helpful when sketching a graph of a function. Other times it can be a crutch. When using a graphing calculator, we must always make sure that an appropriate window is chosen which includes all of the interesting behavior of a graph - otherwise information may be lost. That means to guarantee that we have a good accurate graph of a function $f(x)$ on a calculator, we need to examine the properties of the graph using calculus to make sure they appear on the graph. We illustrate with an example.

Example 5.1. Use a calculator to sketch the graph of

$$f(x) = \frac{e^x}{x^2 - 9}$$

making sure to include all interesting points on the graph.

If we simply sketch $f(x)$ on a standard window on a calculator, we get the following:



Immediately looking at this graph, we know it cannot possibly contain all information since we know there is a vertical asymptote at $x = -3$ which is not illustrated in this picture. Therefore, to check we have all the important parts of the graph, we need to take the steps we outlined previously.

(i) The domain of $f(x)$ is all real numbers except $x = \pm 3$.

(ii) The x intercepts are when $f(x) = 0$ and this never happens. The y-intercept occurs at $y = -1/9$.

(iii) The function is not symmetric in any way.

(iv) There are two vertical asymptotes - one at $x = 3$ and one at $x = -3$. Evaluating one sided limits, we have

$$\lim_{x \to -3^-} f(x) = \infty \text{ and } \lim_{x \to -3^+} f(x) = -\infty$$

and

$$\lim_{x \to 3^-} f(x) = -\infty \text{ and } \lim_{x \to 3+} f(x) = \infty$$

. Also, we have

$$\lim_{x \to \infty} f(x) = \infty \text{ and } \lim_{x \to -\infty} f(x) = 0$$

using L'Hopitals rule.

(v) We have

$$f'(x) = \frac{(x^2 - 2x - 9)e^x}{(x^2 - 9)^2}.$$

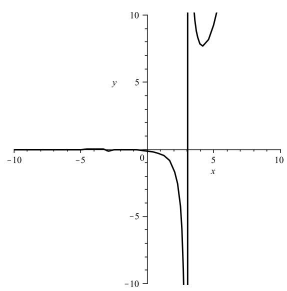
This is undefined at $x = \pm 3$ and equal to 0 when $x = 1 \pm \sqrt{10}$. This means we need to test on the intervals $(-\infty, -3)$, $(-3, 1 - \sqrt{10})$, $(1 - \sqrt{10}, 3)$, $(3, 1 + \sqrt{10})$, $(1 + \sqrt{10}, \infty)$. We have

Interval	Test Point	Inc/Dec
$(-\infty, -3)$	$f'(-10) > 0$	Inc
$(-3, 1-\sqrt{10})$	$f'(-2.5) > 0$	Inc
$(1-\sqrt{10}, 3)$	$f'(0) < 0$	Dec
$(3, 1+\sqrt{10})$	$f'(4) < 0$	Dec
$(1+\sqrt{10}, \infty)$	$f'(10) > 0$	Inc

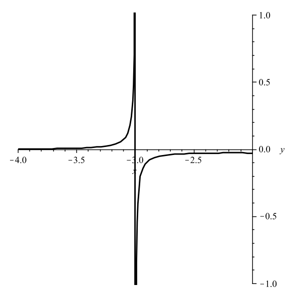
This means that there is a maximum at $x = 1 - \sqrt{10}$ and a minimum at $x = 1 + \sqrt{10}$.

(vi) The second derivative is extremely complicated, so we shall use our calculator to finish off graphing.

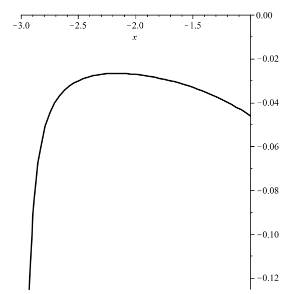
Putting all this information together, we know that from our original graph on the calculator, there are two missing things - the maximum value and the asymptote. To fill these in, we need more than one graph - specifically, we need three - one to give the general shape, one to give the behavior near the asymptote and one near the maximum:



General Graph



Asymptote



Maximum