Sections 4.4: Indeterminate Forms and L'Hopitals Rule

In this section, we return to the idea of evaluating limits. Specifically, we shall develop a rule which allows us to calculate certain special types of limits using derivatives.

1. L'Hopitals Rule

Recall that the first limit we considered was

$$\lim_{x \to 0} \frac{\sin(x)}{x}.$$

We evaluated this by taking values of x closer and closer to $x = 0$ and we concluded that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

This however did not actually prove that this was the limit. Indeed, there is no reason why as we take values closer to 0 than we did, the values of $\sin(x)/x$ do not differ from 1 i.e. however "close" we chose x to be to 0, there are still infinitely may values of x closer. Therefore, we need a better way to determine such limits. Before we do this, we give this type of limit a special name.

Definition 1.1. Suppose that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0.$$

Then we call the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

an indeterminate form of type $0/0$.

The reason we specify the type is that there are other indeterminate forms.

Definition 1.2. Suppose that

$$\lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty.$$

Then we call the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

an indeterminate form of type ∞/∞.

To determine limits of indeterminate forms, we can use the following:

Result 1.3. (L'Hopitals Rule) Suppose that f and g are differentiable functions on an open interval I which contains a and the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

is an indeterminate form of type $0/0$ or ∞/∞. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided this limit exists.

Before we look at examples and explain why this is true, we make a couple of observations:

(*i*) L'Hopitals rule can only be used if all conditions are met i.e. differentiable and indeterminate form.

(*ii*) L'Hopitals rule gives us a new way to calculate the limit of a quotient of functions using derivatives. It DOES NOT give us a way to differentiate. Do not confuse it with the quotient rule - you only apply L'Hopitals rule when finding the limit of an indeterminate form - if you want to differentiate a quotient, use the quotient rule.

(*iii*) L'Hopitals rule works for one-sided limits.

(*iv*) L'Hopitals rule works for limits at infinity.

(*v*) L'Hoptials rule can be applied any number of times provided in each step, the new quotient is an indeterminate form i.e.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} = \lim_{x\to a}\frac{f'''(x)}{g'''(x)} = \dots$$

provided each term is an indeterminate form.

We illustrate with a couple of easy examples.

Example 1.4. Evaluate the following limits.

(*i*)

$$\lim_{x\to 0}\frac{\sin(x)}{x}.$$

This is an indeterminate form of type $0/0$ so we can apply L'Hopitals rule. We have

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{x\to 0}\frac{\cos(x)}{1} = 1$$

and thus

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1.$$

(*ii*)

$$\lim_{x\to 0}\frac{5x+e^{-x}}{7x}.$$

This is an indeterminate form of type ∞/∞ so we can apply L'Hopitals rule. We have

$$\lim_{x\to 0}\frac{5x+e^{-x}}{7x} = \lim_{x\to 0}\frac{5-e^{-x}}{7} = \frac{5}{7}$$

and thus
$$\lim_{x\to 0} \frac{5x + e^{-x}}{7x} = \frac{5}{7}.$$

2. Why L'Hopitals Rule Works and more Examples

Before we look at any more examples, the obvious question to ask is why L'Hopitals rule works. The reason it works is fairly simple and comes down to linear approximations. We shall examine in detail an indeterminate form of type 0/0.

Suppose that $f(a) = 0$ and $g(a) = 0$. Then the linear approximations to $f(x)$ and $g(x)$ at $x = a$ are
$$l_1(x) = f'(a)(x-a) + f(a) \text{ and } l_2(x) = g'(a)(x-a) + g(a).$$
Since $f(a) = g(a) = 0$, we have
$$l_1(x) = f'(a)(x-a) \text{ and } l_2(x) = g'(a)(x-a).$$
Then since the linear approximation of a function at $x = a$ approximates values close to $x = a$, we have
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{l_1(x)}{l_2(x)} = \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$
provided $g'(a) \neq 0$.

We look at some more examples.

Example 2.1. Evaluate the following limits.

(*i*)
$$\lim_{x\to 0} \frac{x^2}{\sin(x)}.$$

This is an indeterminate form of type 0/0 so we can apply L'Hopitals rule. We have
$$\lim_{x\to 0} \frac{x^2}{\sin(x)} = \lim_{x\to 0} \frac{2x}{\cos(x)} = 0$$
and thus
$$\lim_{x\to 0} \frac{x^2}{\sin(x)} = 0.$$

(*ii*)
$$\lim_{x\to 1} \frac{\ln(x)}{x^2 - 1}.$$

This is an indeterminate form of type 0/0 so we can apply L'Hopitals rule. We have
$$\lim_{x\to 1} \frac{\ln(x)}{x^2 - 1} = \lim_{x\to 1} \frac{\frac{1}{x}}{2x} = \frac{1}{2}$$

and thus
$$\lim_{x\to 1} \frac{\ln(x)}{x^2-1} = \frac{1}{2}.$$

(*iii*)
$$\lim_{x\to 0} \frac{\cos(x)}{x}.$$

This is not an indeterminate form since $\cos(0) = 1 \neq 0$. In particular, the limit
$$\lim_{x\to 0} \frac{\cos(x)}{x}$$
does not exist (since it is of the form $1/0$).

(*iv*)
$$\lim_{x\to\infty} \frac{x^2}{e^x}.$$

This is an indeterminate form of type ∞/∞ so we can apply L'Hopitals rule. We have
$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$
applying L'Hopitals rule twice. Thus
$$\lim_{x\to\infty} \lim_{x\to\infty} \frac{x^2}{e^x} = 0.$$

3. Indeterminate Products, Powers and Sums

L'Hopitals rule can be used for other types of indeterminate forms when coupled with some simple algebra. We briefly run over the different types.

3.1. **Indeterminate Products.** If
$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = \infty$$
then we call the product
$$\lim_{x\to a} f(x)\cdot g(x)$$
an indeterminate product. We can evaluate an indeterminate product by writing it as
$$\lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}}$$
or
$$\lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}}$$
and treating it as an indeterminate form of the type $0/0$ or ∞/∞ respectively.

Example 3.1. Evaluate the following limit

$$\lim_{x\to 0} xe^{-x}.$$

This is an indeterminate product. We have

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

using L'Hoptials rule on the indeterminate form x/e^x of the type ∞/∞.

3.2. **Indeterminate Powers.** Any limit of the form

$$\lim_{x\to a} f(x)^{g(x)}$$

of type 1^∞, ∞^0 or 0^0 is called an indeterminate power. To find an indeterminate power, we take its logarithm and convert it into an indeterminate form on which we can use L'Hopitals rule. Once we evaluate this limit, say it is L, then the limit of the indeterminate power will be e^L.

Example 3.2. Evaluate the limit

$$\lim_{x\to 0}(e^x + x)^{\frac{1}{x}}$$

This is an indeterminate power of type 1^∞. We have

$$\lim_{x\to 0} \ln{(e^x + x)^{\frac{1}{x}}} = \lim_{x\to 0} \frac{\ln{(x+e^x)}}{x} = \lim_{x\to 0} \frac{\frac{1}{x+e^x}\cdot(1+e^x)}{1} = 2$$

using L'Hopitals Rule. Thus we have

$$\lim_{x\to 0}(e^x + x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\ln{(e^x+x)^{\frac{1}{x}}}} = e^2.$$

3.3. **Indeterminate Differences.** If

$$\lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to a} g(x) = \infty$$

then we call the difference

$$\lim_{x\to a} f(x) - g(x)$$

an indeterminate difference. We can use L'Hopitals rule to evaluate an indeterminate difference if we can modify it algebraically so it is equivalent to an indeterminate form of one of the types we can evaluate.

Example 3.3. Evaluate the following limit

$$\lim_{x\to 1} \frac{1}{\ln{(x)}} - \frac{1}{x-1}.$$

This is an indeterminate difference. We have

$$\lim_{x\to 1} \frac{1}{\ln{(x)}} - \frac{1}{x-1} = \lim_{x\to 1} \frac{x-1-\ln{(x)}}{(x-1)\ln{(x)}}.$$

This is now an indeterminate form of type 0/0, so we can evaluate it using L'Hopitals rule.

$$\lim_{x\to 1} \frac{x-1-\ln(x)}{(x-1)\ln(x)} = \lim_{x\to 1} \frac{1-\frac{1}{x}}{\ln(x)+(x-1)\frac{1}{x}} = \lim_{x\to 1} \frac{1-\frac{1}{x}}{\ln(x)+1-\frac{1}{x}}$$

$$= \lim_{x\to 1} \frac{\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}} = \frac{1}{2}$$

using L'Hoptials rule twice. Thus

$$\lim_{x\to 1} \frac{1}{\ln(x)} - \frac{1}{x-1} = \frac{1}{2}.$$