Sections 4.7: Optimization

In previous sections we used the derivative to determine minimum and maximum values for a function. In real life, there are many different situations in which this skill could be extremely useful with examples running over may different disciplines. For example, optimizing profit in a business adventure, minimizing fuel required to move an object and so on. In this section we shall examine the problem of minimizing and maximizing quantities in real life situtations (or mathematical models as we shall usually call them.

1. An Example

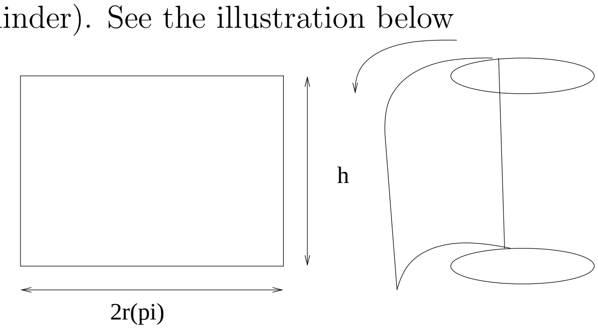
Before we discuss more generally how to solve optimization problems, we shall examine a specific problem.

Example 1.1. (The Coke Can Problem) Determine the optimal dimensions to minimize the surface area of a cylindrical can which holds $355cm^3$.

In order to tackle this problem, we need to first make a note of the information we are given. We are asked to minimize surface area, so we shall need to write down a formula for surface area. Also, we are told that the volume is fixed, so we should write down a formula for volume. Specifically, if the can has height h and radius r, V denotes surface area and SA surface area, then

$$V = \pi r^2 h \text{ and } SA = 2\pi r^2 + 2\pi r h.$$

Note that the formula for surface area was obtained by taking the area of the disc on the top and bottom (both πr^2) and the area of the cylindrical part (which is a rectangle of length $2\pi r$ and height h rolled up into a cylinder). See the illustration below



Since we are trying to minimize surface area, we shall need to take the derivative of the surface area equation and solve for a min. However, the equation is with respect to two different variables, so we cannot differentiate it until one of the variables is eliminated. This can be done by using the volume formula. Specifically, we know that $V = 355$, so it follows that

$$V = \pi r^2 h = 355 \text{ or } h = \frac{355}{\pi r^2}.$$

Thus we have

$$SA = 2\pi r^2 + 2\pi r \frac{355}{\pi r^2} = 2\pi r^2 + \frac{710}{r^2}.$$

Now we have surface area as a function of a single variable, we can use the results of Section 4.1 to determine the minimum value.
The minimum value of SA will either occur at one of the endpoints of the domain of definition or at one of the critical points. First note that the domain of this function will be $(0, \infty)$ (the physical domain). Clearly we cannot have $r = 0$ (since this would be a really tall can with no radius and thus would not hold any liquid). We also could not have $r = \infty$ (since this would be a flat can with no height and this would not hold any liquid). This means the minimum must take place at a critical point. Solving for critical points, we have

$$SA' = 4\pi r - \frac{710}{r^2} = 0 \text{ giving } 4\pi r = \frac{710}{r^2}.$$

Thus we get

$$r^3 = \frac{710}{4\pi} \text{ or } r = 3.837cm.$$

Using the second derivative to check that this is indeed a minimum, we have

$$SA'' = 4\pi + \frac{1420}{r^2} > 0$$

and thus it is a minimum. Thus the dimensions minimizing the surface area would be $r = 3.837cm$ and $h = 7.675$ (using the formula we previously derived to calculate h).
Notice that these are not the dimensions that coke and pepsi use!

2. General Techniques for Optimization Problems

We can summarize the necessary steps we needed to take to solve the previous problem so that we can follow them for other optimization problems. In general, we should try to do the following.

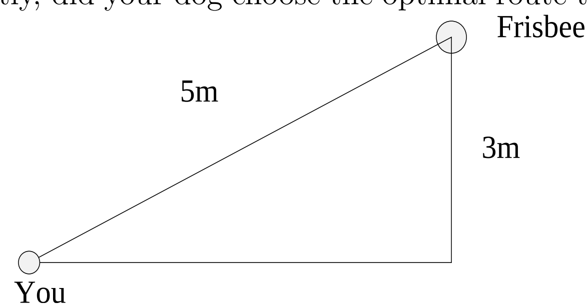
(i) Be clear you know the quantity or function you are trying to optimize - write it down so you do not forget or accidently optimize the wrong function.
(ii) Identify all constants and variables required for the problem and write down all relevant equations relating them.
(iii) Using simple algebra (elimination), try to find a formula for the quantity you are trying to optimize in terms of a single variable. If your formula is in terms of more than one variable, you cannot finish the problem (you will need to do further algebra to eliminate one of the variables).
(iv) Once you have the function you are trying to optimize in terms of a single variable, identify the domain.

(v) Evaluate the critical points and endpoints to locate the minimum and maximum values of the function. If need be, you can check with one of the derivative tests (though if there are no infinite limits, the min will be the smallest and max will be the largest).

Since this topic is one of the more complicated of all of the topics we will cover, we will do lots more examples to illustrate how to tackle these problems.

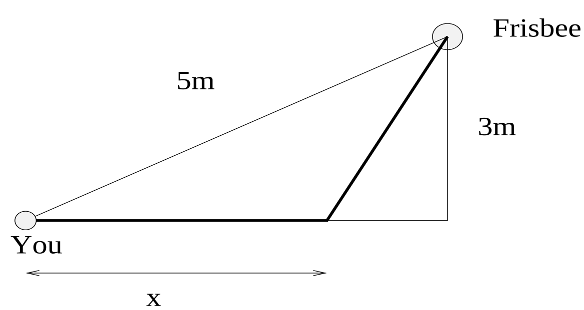
3. Further Optimization Examples

Example 3.1. (Do Dogs Know Calculus?) Suppose you are playing frisbee with your dog on the beach. You stand at a point just touching the shore and throw the frisbee a total distance of $5m$ and it lands $3m$ in from the shoreline (see illustration below). Your dog swims at a rate of $0.5m/s$ and runs at a rate of $3m/s$. If it takes your dog $7.25s$ to retrieve the frisbee, can you conclude that your dog knows calculus? (or equivalently, did your dog choose the optimal route to the frisbee?)



We are trying to determine if the dog traveled took the least amount of time to reach the frisbee, or equivalently, if the dog to the optimum path to minimize time. This means that we need to determine a formula for time traveled to solve the problem. First we need to introduce the relevant variables.

Note that the dog will spend some time running on the beach and some time swimming in the sea. Therefore, let x be the distance the dog travels on ground. Then the dog will take a route as illustrated below.



Using Pythagoras, the distance in the ocean will be

$$\sqrt{9 + (4 - x)^2} = \sqrt{x^2 - 8x + 25}.$$

Since the dog travels $3m/s$ on land, it will spend a total of $x/3$ seconds on land, and since the dog swims at a rate of $0.5m/s$, it will spend a total time of $\sqrt{x^2-8x+25}/0.5 = 2\sqrt{x^2-8x+25}$ seconds in the sea. Thus the total time the dog will spend retrieving the frisbee will be

$$T(x) = \frac{x}{3} + 2\sqrt{x^2-8x+25}.$$

The domain of T will be $0 \leqslant x \leqslant 4$ (the maximum and minimum distances the dog could spend running to retrieve the frisbee). Therefore, we are trying to maximize the function $T(x)$ on the interval $[0,4]$. Since the interval is closed and since $T(x)$ is continuous on this interval, it follows that the global minimum (and maximum) will occur at either one of the endpoints or at a critical point, so we need to evaluate $T(x)$ at each of these points.
We have

$$T'(x) = \frac{1}{3} + \frac{2x-8}{\sqrt{x^2-8x+25}}.$$

This is defined for all x (since $x^2-8x+25 = 9+(4-x)^2 > 0$), so the critical points occur when $T'(x) = 0$. Thus we have

$$\frac{1}{3} + \frac{2x-8}{\sqrt{x^2-8x+25}} = 0 \text{ or } \sqrt{x^2-8x+25} = -6x+24.$$

Squaring both sides, we have

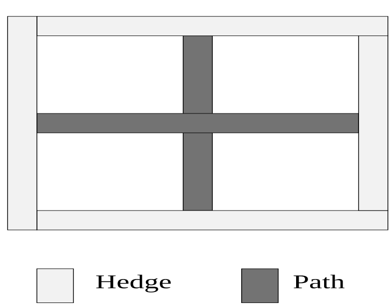
$$x^2-8x+25 = 36x^2-288x+576 \text{ or } 35x^2-280x+551 = 0.$$

Using the quadratic formula, we get $x = 4.507$ or 3.493. Since $x = 4.507$ is not in the physical domain, the only critical point is at $x = 3.493$. Calculating T at each the critical point and the endpoints, we get

$$T(0) = 10, T(4) \sim 7.33, T(3.493) \sim 7.25.$$

Thus the minimum amount of time required for the dog to retrieve the frisbee is $7.25s$ and thus we can conclude that the dog must know calculus (since it optimized its path).

Example 3.2. You are building a garden and you want to minimize the cost to build the garden. The shape of the garden is a rectangle with hedges bordering the edges and two paths cutting, one length ways half way across the garden and the other width ways halfway across the garden (see illustration).



If it costs \$20 per square foot for the path, \$5 per square foot for the hedges and the total area of the garden is $200ft^2$, what are the dimensions required to minimize the cost of the garden?

We need to determine a formula for the cost of the garden. The area is fixed, but we are able to change the width and the length of the garden. Let w denote the l is the length of the garden and w the width. Note that the cost to build the path will be $20(l+w)$ and the cost to build the hedge will be $2(5l+5w)$, so the total cost C will be

$$C = 30l + 30w.$$

We want to minimize this function, but it is expressed in terms of two different variables, so we need to eliminate one. However, since the area is fixed, we also know that

$$l \cdot w = 200 \text{ or } l = \frac{200}{w}$$

and thus

$$C(w) = \frac{6000}{w} + 30w.$$

The domain of this function will be $(0, \infty)$. To determine minimum and maximum values, we find the critcal points. We have

$$C'(w) = -\frac{6000}{w^2} + 30.$$

This is defined on the domain of $C(w)$, so the only critical points occur when $C'(w) = 0$, or

$$-\frac{6000}{w^2} + 30 \text{ giving } w^2 = 200$$

or $w = 10\sqrt{2}$ giving also $l = 10\sqrt{2}$. To check that this is indeed a minimum, we need to check the endpoints. However, observe that

$$\lim_{w \to 0} C(w) = \infty \text{ and } \lim_{w \to \infty} C(w) = \infty$$

and thus this must indeed be a minimum value. Therefore, the cheapest dimensions are $10\sqrt{2} \times 10\sqrt{2}$ (a square). It follows that the minimal cost will be

$$C(10\sqrt{2}) = 600\sqrt{2} \sim \$849.$$

Example 3.3. A wire of length $10cm$ is snapped into two pieces. One is bent into the shape of an equilateral triangle and the other into a circle. Find where to break it to maximize the area bounded by these two shapes.

In this case, we are looking at where to snap the wire to optimize area bounded by these two shapes, so let x denote the length of wire we use to make a triangle. Then $10 - x$ will be the amount of wire used to make the circle. Next we need to calculate area.

Since we are using $x$cm of wire to make the circle, that will equal the circumference of the circle we are trying to make i.e. $10 - x = 2\pi r$. This means $r = (10 - x)/2\pi$, and thus the area of the circle will be

$$A_C = \pi r^2 = \pi \frac{(10-x)^2}{4\pi^2} = \frac{x^2 - 20x + 100}{4\pi}.$$

The length of the wire to make the triangle is x, so each side length will be $x/3$. Since the area of an equilateral triangle with side length s is $\frac{\sqrt{3}s^2}{4}$, we have

$$A_T = \frac{\sqrt{3}x^2}{36}.$$

Therefore, the total area will be

$$A(x) = \frac{x^2 - 20x + 100}{4\pi} + \frac{\sqrt{3}x^2}{36}.$$

The derivative is

$$A'(x) = \frac{2x - 10}{4\pi} + \frac{\sqrt{3}x}{18}.$$

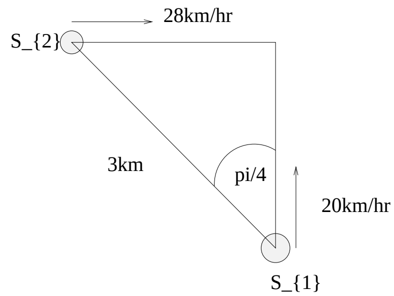
This is defined for all values of x, and this the only critical points occur when $A'(x) = 0$. Setting $A'(x) = 0$ and solving for x, we get $x \sim 3.12$. To determine the global min and max, we note that the domain of $A(x)$ is $0 \leqslant x \leqslant 10$, so we are trying to maximize $A(x)$ on the closed interval $[0, 10]$. Since $A(x)$ is continuous, it follows that a global min and max must be attained at either a critical point or an endpoint. Plugging in the values, we get

$$A(0) \sim 7.95, A(10) = 4.811, A(3.12) = 4.24.$$

It follows that the maximum value occurs at the endpoint $x = 0$ i.e. when we only construct a circle.

Example 3.4. A ship S_1 is traveling north at $20km/hr$. Another ship, S_2 is traveling east at $28km/hr$. Initially S_2 is $3km$ northwest of S_1. For safety reasons, they cannot get within $100m$ of each other. Will one of the ships have to change course?

We need to minimize the distance between the two ships to determine whether or not the distance between them could drop below $100m$. We shall calculate the distance between the two ships as a function of time. Suppose that the point of intersection of the two paths of the ships is the origin and suppose at $t = 0$ they are $3km$ from each other (as given in the statement of the problem). Also note that the angle between the ships initially is $\pi/4$ (since one ship is due northwest of the other). Then we have the following set up:



Since the interior angle is $\pi/4$ and it is a right triangle, we can determine the distance of each ship from the origin. Specifically, they will both start an initial distance of $3\sqrt{2}/2$ away from the point of intersection. Then, since they are both travel at constant speeds, the distance of each from the origin will be easy to calculate. Specifically, if D_1 denotes the distance of S_1 from the origin and D_2 the distance of S_2 from the origin, we have

$$D_1 = \frac{3\sqrt{2}}{2} - 20t \text{ and } D_2 = \frac{3\sqrt{2}}{2} - 28t.$$

Thus the total distance between S_1 and S_2 will be

$$D = \sqrt{D_1^2 + D_2^2} = \sqrt{1184t^2 - 124t + \frac{18}{2}}$$

after simplification. Since this is a square root which will always be positive, in order to minimize or maximize it, we can simply minimize or maximize what is inside the square root i.e. $f(t) = 1184t^2 - 124t + \frac{18}{2}$. The critical points of this functions are when $f'(t) = 2368t - 124 = 0$ or $t \sim 0.05$. Since the domain of D is $[0, \infty)$ and we know $\lim_{t\to\infty} D(t) = \infty$, it follows that there will be a minimim value of $D(t)$ either at the critical point $t = 0.05$ or at the endpoint $t = 0$. Plugging in, we get $D(0.05) = 2.4km$ and $D(0) = 3$. Thus the minimum distance between the two ships is $2.4km$ so they will not even come close to touching (so they do not need to change course).