# Sections 4.9: Antiderivatives

For the last few sections, we have been developing the notion of the derivative and determining rules which allow us to differentiate all different types of functions. In this section, and for the remainder of the semester, we shall consider the reversal of finding a derivative. Specifically, give a function $f(x)$, we shall consider methods to determine how to find a function $F(x)$ whose derivative is $f(x)$ i.e. $F'(x) = f(x)$. Such techniques will be useful in modeling many different things such as distance given velocity.

## 1. Antiderivatives

We start with a definition.

**Definition 1.1.** A function $F(x)$ is called an antiderivative of $f(x)$ if $F'(x) = f(x)$.

The first obvious question to ask is given a function $f(x)$, how many antiderivatives of $f(x)$ are there. We considered this question when we were studying the mean value theorem, and we derived the following answer:

**Result 1.2.** Suppose $F(x)$ and $G(x)$ are antiderivatives of a function $f(x)$. Then $F(x) = G(x) + C$ for some constant $C$. That is, any two antiderivatives of a function $f(x)$ differ by a constant.

Also note that if $F(x) = G(x) + C$, then $F'(x) = G'(x)$. With this is mind, when determining antiderivatives, we usually try to write down the most general antiderivative unless we are specifically asked to specify a particular antiderivative. Therefore, we define the following.

**Definition 1.3.** We define the most general antiderivative of $f(x)$ to be $F(x) + C$ where $F'(x) = f(x)$ and $C$ represents an arbitrary constant. If we choose a value for $C$, then $F(x) + C$ is a specific antiderivative (or simply an antiderivative of $f(x)$).

We consider some examples.

**Example 1.4.** Find the most general antiderivatives of the following:

*(i)*

$$f(x) = x^2$$

The most general antiderivative is $F(x) = x^3/3 + C$ for an arbitrary constant $C$.

*(ii)*

$$g(x) = \frac{5 - 4x^3 + 2x^6}{x^6}$$

First, simplifying, we get

$$g(x) = \frac{5 - 4x^3 + 2x^6}{x^6} = \frac{5}{x^6} - \frac{4}{x^3} + 2 = 5x^{-6} - 4x^{-3} + 2.$$

The most general antiderivative is

$$G(x) = -x^{-5} + 2x^{-2} + 2x + C$$

for an arbitrary constant $C$.

*(iii)*

$$k(x) = 3x + 2x^{1.7}$$

The most general antiderivative is

$$K(x) = \frac{3x^2}{2} + \frac{2x^{2.7}}{2.7} + C$$

for an arbitrary constant $C$.

*(iv)*

$$f(x) = 2x + 5(1 - x^2)^{-\frac{1}{2}}$$

The most general antiderivative is

$$F(x) = x^2 + 5\arcsin(x) + C$$

for an arbitrary constant $C$ since the derivative of $\arcsin(x)$ is $(1 - x^2)^{-1/2}$.

**Example 1.5.** *(i)* Find $F(x)$ if

$$F'(x) = 4 - 3(1 + x^2)^{-1}$$

and $F(1) = 0$.

The most general antiderivative is

$$F(x) = 4x - 3\arctan(x) + C$$

for an arbitrary constant $C$ since the derivative of $\arctan(x)$ is $(1 + x^2)^{-}$. To find the specific antiderivative, we evaluate:

$$F(1) = 4 - 3\arctan(1) + C = 0 \text{ so } C = 3\arctan(1) - 4 = \frac{\pi}{4} - 4.$$

Thus we have

$$F(x) = 4x - 3\arctan(x) + \frac{\pi}{4} - 4.$$

*(ii)* Find $G(x)$ if

$$G'(x) = \sqrt{x}(6 + 5x)$$

and $G(1) = 10$.

Simplifying, we have $G'(x) = 6x^{1/2} + 5x^{3/2}$. The most general antiderivative is

$$G(x) = 4x^{\frac{3}{2}} + 2x^{\frac{5}{2}} + C$$

for an arbitrary constant $C$. To find the specific antiderivative, we evaluate:

$$G(1) = 4 + 2 + C = 10 \text{ so } C = 4.$$

Thus we have

$$G(x) = 4x^{\frac{3}{2}} + 2x^{\frac{5}{2}} + 4.$$

**Example 1.6.** Find $f(x)$ given that $f''(x) = 3/\sqrt{x}$, $f(4) = 20$ and $f'(4) = 7$.

The most general antiderivative of $f''(x)$ is

$$f'(x) = 6x^{\frac{1}{2}} + C$$

for an arbitrary constant $C$. To find the specific antiderivative, we evaluate:

$$f'(4) = 6 \cdot 2 + C = 7 \text{ so } C = -1.$$

Thus we have

$$f'(x) = 6x^{\frac{1}{2}} - 1.$$

Next, the most general antiderivative of

$$f'(x) = 6x^{\frac{1}{2}} - 1$$

is

$$f(x) = 4x^{\frac{3}{2}} - x + C$$

for an arbitrary constant $C$. To find the specific antiderivative, we evaluate:

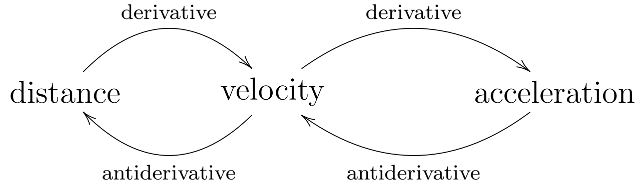
$$f(4) = 4 \cdot 8 - 4 + C = 20 \text{ so } C = 0.$$

Thus we have

$$f(x) = 4x^{\frac{3}{2}} - x.$$

## 2. Rectilinear Motion

As mentioned earlier, one of the applications of antiderivatives is that of the motion of an object moving in a straight line. Since distance, velocity and acceleration are all related by derivatives, they are also related by antiderivatives (see below).




$s(t)$ $\qquad$ $v(t) = s'(t)$ $\qquad$ $a(t) = v'(t) = s''(t)$

We illustrate the types of problems which can be solved using this knowledge with some explicit examples.

**Example 2.1.** If you are given the following information about the movement of a particle at time $t$, find a formula for its position:

$$a(t) = t - 2, s(0) = 1 \text{ and } v(0) = 3.$$

Since $a(t) = t - 2$, we have

$$v(t) = \frac{t^2}{2} - 2t + C$$

where $C$ is an arbitrary constant. To find $C$ we evaluate getting

$$v(0) = C = 3.$$

Thus we have

$$v(t) = \frac{t^2}{2} - 2t + 3.$$

Next we calculate

$$s(t) = \frac{t^3}{6} - t^2 + 3t + C.$$

To determine $C$, again we evaluate getting

$$s(0) = C = 1$$

and thus

$$s(t) = \frac{t^3}{6} - t^2 + 3t + 1.$$

**Example 2.2.** A stone is dropped from a cliff and hits the ground at $120ft/s$. What is the height of the cliff? (note that the acceleration due to gravity is $a(t) = -32ft/s^2$).

We have $a(t) = -32$, and so

$$v(t) = -32t + C$$

where $C$ is an arbitrary constant. To find $C$ we evaluate getting

$$v(0) = C = 0$$

since the initial velocity is 0. Thus we have

$$v(t) = -32t.$$

Next we calculate

$$s(t) = -16t^2 + C.$$

Note that when $t = 0$ we have $s(0) = C$, and so $C$ is the height of the cliff. In particular, to answer the question, we need to determine $C$. We know that when the stone hits the ground, we have

$$v(t) = -32t = -120$$

and so it follows that the rock hits the ground when $t = 120/32 = 15/4$. Since the rock will have height 0 when it hits the ground, we have

$$s\left(\frac{15}{4}\right) = -16 \cdot \frac{225}{16} + C = 0$$

and thus

$$C = 225.$$

Thus the height of the cliff is $250ft$.