Sections 5.1: Areas and Distances

In this section we shall consider problems closely related to the problems we considered at the beginning of the semester (the tangent and velocity problems). Specifically, we shall consider the problem of finding the area of the region bounding by a function $y = f(x)$ and the x-axis, and we shall consider the problem of determining the distance traveled by a car if we know its velocity.

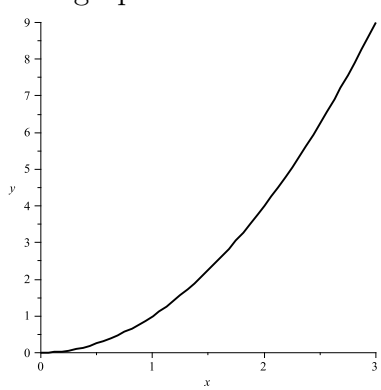
1. The Area Problem

We start with a simple question.

Question 1.1. Suppose that $f(x)$ is a continuous function on the interval $[a, b]$ and $f(x) \geqslant 0$. How can we calculate the area bounded between $f(x)$ and the x-axis on the interval $[a, b]$?

Answer. We shall approximate the area using rectangles, and then take smaller and smaller rectangles to determine an answer closer and closer to the exact answer. We illustrate with an example.

Example 1.2. Suppose we are trying to estimate the area under $y = x^2$ on the interval $[0, 3]$. The graph looks like the following.



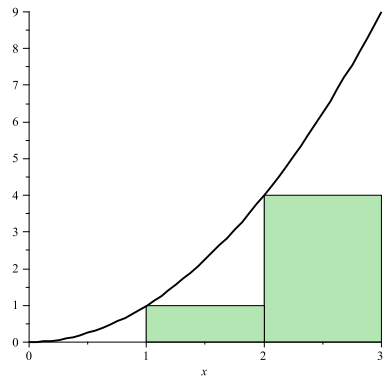
Then we proceed as follows:

(i) First note that since the graphs is curved, we cannot use elementary geometry to calculate the area. However, we can use elementary geometry to *estimate* the area between the curves using rectangles.

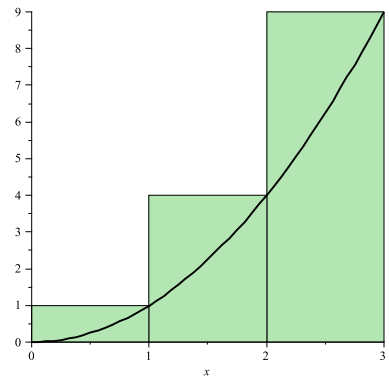
(ii) We start by breaking up the interval $[0, 3]$ into 3 equal sized pieces of size $\Delta x = (3 - 0)/3 = 1$.

(iii) We can approximate the area between $f(x)$ and the x-axis in each of these subintervals by using the rectangle whose height is determined by the value of $f(x)$ at one of the two endpoints of each subinterval. Specifically, taking left hand endpoints, in the 1st, we approximate the area with a rectangle whose top edge has height $f(0) = 0$ and whose length is $\Delta x = 1$, in

the 2nd, we approximate the area with a rectangle whose top edge has height $f(1) = 1$ and whose length is $\Delta x = 1$, and in the 3rd, we approximate the area with a rectangle whose top edge has height $f(2) = 4$ and whose length is $\Delta x = 1$. We illustrate with the left hand endpoints and the right and endpoints below:



Left Hand Endpoints



Right Hand Endpoints

(iv) In general, the area of each rectangle will be $f(x)\Delta x$, where $\Delta x = 1$ is the length of each subinterval and x is either the right or left hand endpoint. summing over all rectangles, we get the following two approximations for the area:

$$\text{Left Hand Sum} = (f(0) + f(1) + f(2))\Delta x = 5,$$

$$\text{Right Hand Sum} = (f(1) + f(2) + f(3))\Delta x = 14.$$

Observe that the left hand sum is an underestimate of the actual area, and the right hand sum is an overestimate of the actual area, and thus we have

$$5 \leqslant \text{Area} \leqslant 14.$$

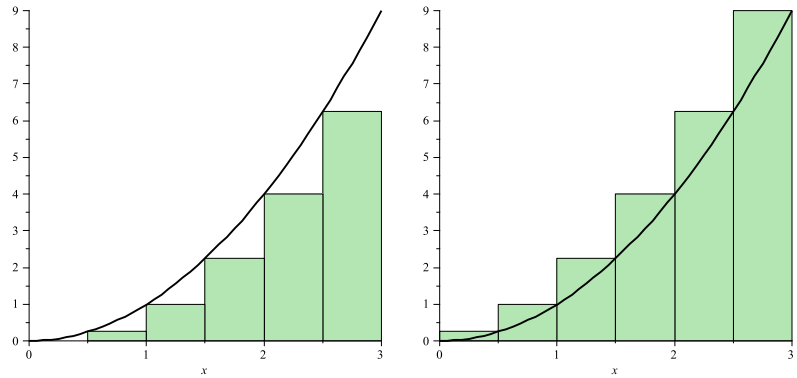
(v) In order to improve our approximation, we can take smaller values of Δx noting that as $\Delta x \to 0$ or equivalently as the number of subintervals n goes to ∞, the answer becomes more exact. We illustrate with the left and right hand sums with $n = 6$. In this case we have

$$\Delta x = \frac{3-0}{6} = \frac{1}{2}.$$

Calculating the two different sums, we have

$$\text{Left Hand Sum} = (f(0)+f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)+f(2)+f\left(\frac{5}{2}\right))\Delta x = 6.875,$$

Right Hand Sum $= (f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)+f(2)+f\left(\frac{5}{2}\right)+f(3))\Delta x = 11.375.$



Left Hand Endpoints Right Hand Endpoints

Geometrically, we can see that the sums are a lot closer to the actual value of the area, but both are still off. Again, since the left hand sum is an underestimate of the actual area, and the right hand sum is an overestimate of the actual area, and thus we have

$$6.875 \leqslant \text{Area} \leqslant 11.375.$$

(*vi*) To make the answer more exact, we can take smaller subdivisions.

Our observations in the last example motivate the following:

Definition 1.3. Suppose that $f(x)$ is a continuous function over an interval $[a, b]$ and $f(x) \geqslant 0$. Let $\Delta x = (b-a)/n$, and if we subdivide $[a, b]$ up into equal spaced subintervals, let $a = x_0 < x_1 < x_2 \cdots < x_n = b$ be the endpoints of these subintervals.

(*i*) We define the nth right hand Riemann sum to be

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = (f(x_1) + f(x_2) + \cdots + f(x_n))\Delta x.$$

(*ii*) We define the nth right hand Riemann sum to be

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x = (f(x_0) + f(x_2) + \cdots + f(x_{n-1}))\Delta x.$$

(*iii*) We define the nth midpoint Riemann sum to be

$$M_n = \sum_{i=1}^{n} f\left(\frac{x_i - x_{i-1}}{2}\right)\Delta x = (f\left(\frac{x_1 - x_0}{2}\right)+f\left(\frac{x_2 - x_1}{2}\right)+\cdots+f\left(\frac{x_n - x_{n-1}}{2}\right))\Delta x.$$

Note that all three of these sums approximate the area of $f(x)$ on the interval $[a, b]$. Specifically, we have the following result.

Result 1.4. Suppose that $f(x)$ is a continuous function over an interval $[a, b]$ and $f(x) \geqslant 0$. Then the area A bounded between $f(x)$ and the x-axis on the interval $[a, b]$ is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} M_n.$$

We illustrate with a couple of examples.

Example 1.5. (i) Explain when you can guarantee that L_n is an underestimate or an overestimate (and similarly for R_n).

For any function which is strictly increasing, R_n will always be an overestimate of area and L_n will always underestimate area. Likewise, for any strictly decreasing function, R_n will always be an underestimate of area and L_n will always overestimate area. If a function is both increasing and decreasing, the we cannot guarantee that either one is an over or underestimate.

(ii) Suppose that values of $f(x)$ are given in the following table. Use it to estimate the area bounded between $f(x)$ and the x-axis on the interval $[1, 3]$.

x	1	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$	$\frac{8}{3}$	3
f	0	2	3	4	3	2	5

We can use left and right hand sums to approximate the area. Specifically, we have $\Delta x = 1/3$, and so

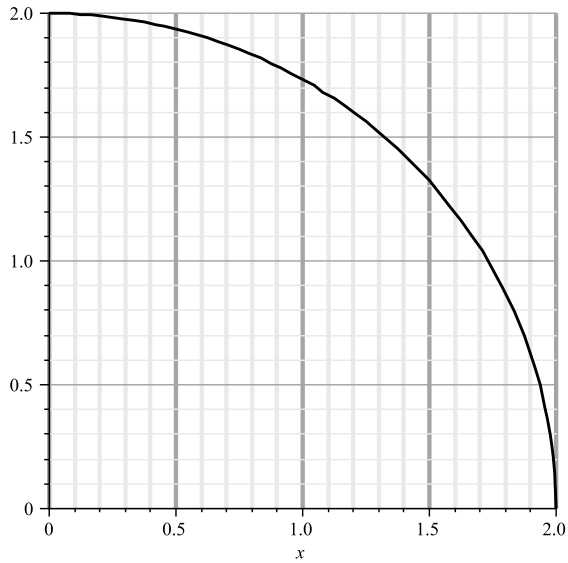
$$\text{Left Hand Sum} = (f(1)+f\left(\frac{4}{3}\right)+f\left(\frac{5}{3}\right)+f(2)+f\left(\frac{7}{3}\right)+f\left(\frac{8}{3}\right))\Delta x$$

$$= (0+2+3+4+3+2)\frac{1}{3} = \frac{14}{3},$$

$$\text{Right Hand Sum} = (\left(\frac{4}{3}\right)+f\left(\frac{5}{3}\right)+f(2)+f\left(\frac{7}{3}\right)+f\left(\frac{8}{3}\right)+f(3))\Delta x$$

$$= (2+3+4+3+2+5)\frac{1}{3} = \frac{19}{3}.$$

(iii) Suppose that the graph below is the graph of $f(x)$. Determine L_4 on the interval $[0, 2]$.



Using the gridlines, we can set up a Riemann sum. Since we are calculating L_4, we have $\Delta x = 1/2$. Estimating the value of the function at each left endpoint, we have $f(0) \sim 2$, $f(\frac{1}{2}) \sim 1.9$, $f(1) \sim 1.7$, and $f(\frac{3}{2}) \sim 1.3$. Thus we have

$$L_4 = (2 + 1.9 + 1.7 + 1.3) \cdot \frac{1}{2} = 3.45.$$

As we observed, the more subintervals we take, the more exact our answer will become. The calculations however are extremely tedious, so when we are calculating large Riemann sums, we usually use a calculator to determine the values. We illustrate.

Example 1.6. Determine the number of subintervals needed to guarantee that L_n is within 0.1 of the actual area bounded by $f(x) = x^2$ on the interval $[0, 3]$.

Since $f(x) = x^2$ is a strictly increasing function, to guarantee that L_n is within 0.1 of the actual answer, we simply need to guarantee that R_n and L_n are within 0.1 of the actual answer. However,

$$|R_n - L_n| = |f(3) - f(0))|\Delta x = (9 - 0)\frac{3}{n} = \frac{27}{n}.$$

Thus we want

$$\frac{27}{n} < 0.1 \text{ or } 270 = \frac{27}{0.1} < n.$$

Using 271 intervals on the calculator, we get that the area is approximately 8.95.

2. The Distance Problem

A closely related problem to the area problem is the distance problem. Specifically, we ask ourselves the following question.

Question 2.1. Given the velocity of a moving object, how do we find the distance it has traveled at any given time t?

We illustrate with an example.

Example 2.2. Suppose that you are measuring the velocity of a moving bike and you record the information in the table below. Use these values to estimate the total distance traveled by the bike.

time t	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
velocity v	0	$\frac{1}{4}$	1	$\frac{9}{2}$	4	$\frac{25}{4}$	9

We first note that if an object is traveling at a fixed velocity v over a period of time t, then $D = v \cdot t$ i.e. the distance traveled equals velocity times time. In this case, the bike is changing speed, so we cannot simply multiply time by velocity. We can however estimate the distance traveled over a given interval. Specifically, between $t = 0$ and $t = \frac{1}{2}$, the bike starts travelling at $0m/s$ and at the end of the interval is traveling at $\frac{1}{2}m/s$. We can therefore estimate the total distance traveled by using one of these values - we use $0m/s$ - using $v = 0$, the distance traveled would be approximately $0 \cdot \frac{1}{2} = 0$. Likewise, we can estimate the distance traveled from $t = \frac{1}{2}$ to $t = 1$ by using the velocity $v = \frac{1}{2}m/s$ (or $1m/s$) - using $v = \frac{1}{2}$, the distance traveled would be approximately $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$. We can continue this process by estimating the distance traveled using the velocity at the left hand endpoint of each time interval and multiplying it by the length of the interval (or time elapsed). Likewise, we can do this with the right hand endpoints. We get

$$\text{Left Hand Approximate} = (v(0)+v\left(\frac{1}{2}\right)+v(1)+v\left(\frac{3}{2}\right)+v(2)+v\left(\frac{5}{2}\right))\frac{1}{2}$$

$$= (0 + \frac{1}{4} + 1 + \frac{9}{4} + 4 + \frac{25}{4})\frac{1}{2} = 6.875m$$

$$\text{Right Hand Approximate} = (v\left(\frac{1}{2}\right)+v(1)+v\left(\frac{3}{2}\right)+v(2)+v\left(\frac{5}{2}\right)+v(3))\frac{1}{2}$$

$$= (\frac{1}{4} + 1 + \frac{9}{4} + 4 + \frac{25}{4} + 9)\frac{1}{2} = 11.375m.$$

Thus the total distance is somewhere between $6.875m$ and $11.375m$.

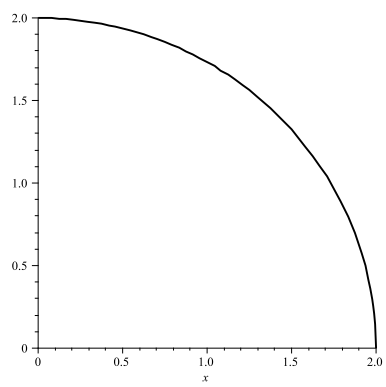
The answers we obtained in the previous example were probably fairly inaccurate as an approximation, so to improve our approximation, we could increase the number of measurements (or decrease the size of the intervals we are taking measurements over). Note however that this is precisely what we did when determining area i.e. finding area and distance traveled are essentially the same problem. We summarize.

Result 2.3. Suppose that the velocity of an object is given by $v(t)$. Then the distance traveled by the object for $a \leqslant t \leqslant b$ is equal to the area under the graph of $v(t)$ on the interval $[a, b]$. Alternatively, the distance D is equal to

$$D = \sum_{i=1}^{n} v(t_i)\Delta t$$

where $\Delta t = (b - a)/n$ and t_i is point in time in the ith time interval (usually the left or right hand time value).

Example 2.4. Suppose the velocity of an object is given by $v(t) = \sqrt{4 - t^2}$ for $0 \leqslant t \leqslant 2$. Find the exact distance traveled by the object during this time.

Since the distance problem is the same as the area problem, we can simply graph $v(t)$ and determine the area under its graph. Observe that the graph is a quarter circle of radius 2, and thus the distance traveled will be

$$\pi(2^2)/4 = \pi.$$