

Sections 5.2: The Definite Integral

In this section we shall formalize the ideas from the last section to functions in general.

1. THE DEFINITE INTEGRAL

We start with a formal definition.

Definition 1.1. Suppose $f(x)$ is a continuous function on the interval $[a, b]$ and let n be some positive integer. Do the following:

- (i) Subdivide the interval $[a, b]$ into n equal sized pieces, each of length $\Delta x = (b - a)/n$. Let the endpoints of these intervals be $a = x_0 < x_1 < x_2 < \dots < x_n = b$.
- (ii) From the i th subinterval $[x_{i-1}, x_i]$, choose a reference point, call it x_i^* (do this for each subinterval).

We define the definite integral of $f(x)$ from a to b to be

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$$

provided this limit exists. If this limit exists, we say that $f(x)$ is integrable on $[a, b]$.

Before we start looking at examples of how to calculate the definite integral, we need some terminology and some basic facts.

Terminology. Suppose that

$$\int_a^b f(x)dx$$

is a definite integral.

- (i) We call $f(x)$ the integrand.
- (ii) We call a and b the limits of integration - a the lower limit and b the upper limit.
- (iii) The procedure of finding the limit is called integration.
- (iv) The sum is called a Riemann sum.

Fact. Suppose that

$$\int_a^b f(x)dx$$

is a definite integral.

- (i) If $f(x) \geq 0$ on $[a, b]$, then

$$\int_a^b f(x)dx$$

measures the area bounded between $f(x)$ and the x -axis between a and b .

(ii) In general,

$$\int_a^b f(x)dx$$

measures the weighted area bounded between $f(x)$ and the x -axis between a and b - by weighted we mean that all area below the x -axis is counted negatively and all area above the x -axis is counted positively.

(iii) If $f(x)$ is continuous on $[a, b]$ except for at finitely many discontinuities, then $f(x)$ is integrable on $[a, b]$.

This last observation means that nearly all functions we shall consider will be integrable. We are now ready to consider some explicit examples. Note that the book emphasizes the use of formulas to calculate integrals - we shall avoid this instead emphasizing on how to calculate them by hand, geometrically and using the calculator.

Example 1.2. Approximate the following integrals using the specified rules.

(i)

$$\int_0^1 (x^2 + 2x)dx$$

with $n = 3$ using the left hand sum.

We have $\Delta x = 1/3$, so

$$\begin{aligned} \text{Left Hand Sum} &= \left(f(0) + f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) \right) \cdot \frac{1}{3} \\ &= \left(0 + \frac{1}{9} + \frac{2}{3} + \frac{4}{9} + \frac{4}{3} \right) \cdot \frac{1}{3} = \frac{23}{27}. \end{aligned}$$

(ii)

$$\int_3^4 \frac{\sin(x)}{x} dx$$

with $n = 2$ using the midpoint sum.

We have $\Delta x = 1/2$, so if $f(x) = \sin(x)/x$ then

$$\begin{aligned} \text{Midpoint Sum} &= \left(f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right) \right) \cdot \frac{1}{2} \\ &= (-0.033291 - 0.152416) \cdot \frac{1}{2} = -0.092854. \end{aligned}$$

(iii)

$$\int_1^3 \frac{\ln(x)}{x} dx$$

with $n = 4$ using the right hand sum.

We have $\Delta x = (3 - 1)/4 = 1/2$, so if $f(x) = \ln(x)/x$ then

$$\begin{aligned} \text{Right Hand Sum} &= \left(f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) + f(3) \right) \cdot \frac{1}{2} \\ &= (0.27 + 0.347 + 0.367 + 0.366) \cdot \frac{1}{2} = 0.675. \end{aligned}$$

Example 1.3. Suppose that the values of $f(x)$ are given in the table below.

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	3	-2	4	5	2	2	-1	-3	-9	0	2

Approximate the following integrals using the specified rules.

(i)

$$\int_0^4 f(x) dx$$

with $n = 4$ using the left hand sum.

We have $\Delta x = 1$, so

$$\begin{aligned} \text{Left Hand Sum} &= (f(0) + f(1) + f(2) + f(3)) \cdot 1 \\ &= (3 - 2 + 4 + 5) \cdot 1 = 10. \end{aligned}$$

(ii)

$$\int_0^8 f(x) dx$$

with $n = 4$ using the midpoint sum.

We have $\Delta x = 2$, so

$$\begin{aligned} \text{Midpoint Sum} &= (f(1) + f(3) + f(5) + f(7)) \cdot 2 \\ &= (-2 + 5 + 2 - 3) \cdot 2 = 4. \end{aligned}$$

(iii)

$$\int_0^8 f(x) dx$$

with $n = 4$ using the right hand sum.

We have $\Delta x = 2$, so

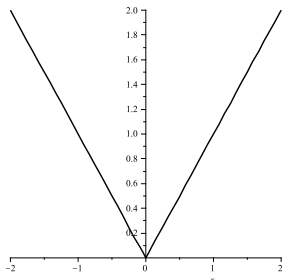
$$\begin{aligned} \text{Right Hand Sum} &= (f(2) + f(4) + f(6) + f(8)) \cdot 2 \\ &= (4 - 1 - 3 - 9) \cdot 2 = -18. \end{aligned}$$

Example 1.4. Evaluate the following integrals exactly.

(i)

$$\int_{-1}^1 |x| dx.$$

First we look at the graph of $f(x)$:



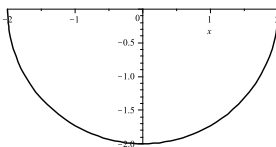
Since the definite integral is equal to the weighted area bounded between the x -axis and the graph of $f(x) = |x|$, we simply need to evaluate the area of the two triangles in the graph. i.e.

$$\int_{-1}^1 |x| dx = 1.$$

(ii)

$$\int_{-2}^0 -\sqrt{4-x^2} dx.$$

As before, we look at the graph of $f(x)$:



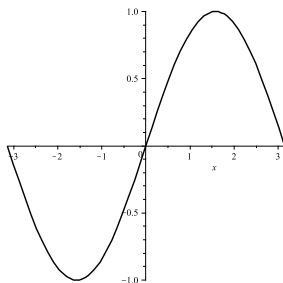
Since the definite integral is equal to the weighted area bounded between the x -axis and the graph of $f(x) = -\sqrt{4-x^2}$, we simply need to evaluate the area of the quarter circle in the third quadrant and negate this value (since it is below the axis). i.e.

$$\int_{-2}^0 -\sqrt{4-x^2} dx = \pi.$$

(iii)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x) dx.$$

As before, we look at the graph of $f(x)$:



Note that between $x = -\pi/2$ and $x = \pi/2$, there is an equal amount of area bounded between the x -axis and $f(x) = \sin(x)$ above and below the x -axis. Since the definite integral is equal to the weighted area bounded between the x -axis it follows that the integral must be zero. i.e.

$$\int_{-\pi/2}^{\pi/2} \sin(x) dx = 0.$$

2. PROPERTIES OF THE DEFINITE INTEGRAL

As with derivatives, there are many useful properties of definite integrals allowing us to calculate them more easily.

Result 2.1. Suppose that $f(x)$ and $g(x)$ are continuous functions on an interval $[a, b]$ and suppose that c is some constant. Then we have the following:

(i)

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

(ii)

$$\int_a^b c dx = c(b - a).$$

(iii)

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

(iv)

$$\int_a^b cf(x) dx = c \int_a^b f(x) dx.$$

(v)

$$\int_a^b (f(x) - g(x)) dx = \int_a^b f(x) dx - \int_a^b g(x) dx.$$

(vi)

$$\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx.$$

(vii)

$$\int_a^b f(x)dx = - \int_b^a f(x)dx.$$

In addition to the basic algebraic properties of integrals, there are a number of methods we can use to help approximate values of integrals. Specifically, we have the following.

Result 2.2. (Comparisons for Integrals)

(i) If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x)dx \geq 0.$$

(ii) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

(iii) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

We finish by illustrating with some examples of how to use these properties and comparisons.

Example 2.3. If

$$\int_0^3 f(x)dx = 2 \text{ and } \int_1^3 f(x)dx = 1$$

find

$$\int_0^1 f(x)dx.$$

We know that

$$\int_0^1 f(x)dx + \int_1^3 f(x)dx = \int_0^3 f(x)dx$$

so

$$\int_0^1 f(x)dx + 1 = 2$$

or

$$\int_0^1 f(x)dx = 1.$$

Example 2.4. Show that

$$\int_0^1 \sin(x)dx \leq \frac{1}{2}$$

by using the fact that $\sin(x) \leq x$ on the interval $[0, 1]$.

Using comparison, we know

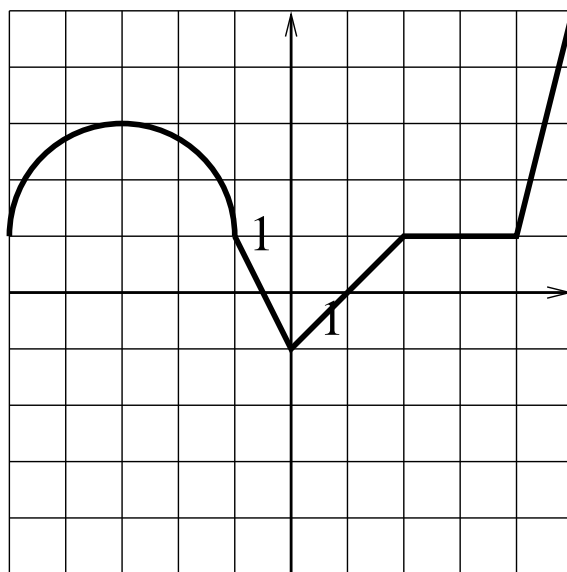
$$\int_0^1 \sin(x) dx \leq \int_0^1 x dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2}.$$

Example 2.5. Fill in the following table for $F(x)$ given that

$$F(x) = \int_0^x f(x) dx$$

where the graph of $f(x)$ is given below.

x	-5	-3	-1	0	1	2	3	4	5
$F(x)$	$-2\pi - 4$	$-\pi - 2$	0	0	$-\frac{1}{2}$	0	1	2	5



We calculate each part individually by determining appropriate areas between the x -axis and the function. We have

$$F(-5) = \int_0^{-5} f(x) dx = - \int_{-5}^0 f(x) dx = - \left(\frac{\pi \cdot 2^2}{2} + 4 + \frac{1}{4} - \frac{1}{4} \right) = -2\pi - 4.$$

$$F(-3) = \int_0^{-3} f(x) dx = - \int_{-3}^0 f(x) dx = - \left(\frac{\pi \cdot 2^2}{4} + 2 + \frac{1}{4} - \frac{1}{4} \right) = -\pi - 2.$$

$$F(-1) = \int_0^{-1} f(x) dx = - \int_{-1}^0 f(x) dx = - \left(\frac{1}{4} - \frac{1}{4} \right) = 0.$$

$$F(0) = \int_0^0 f(x) dx = 0.$$

$$F(1) = \int_0^1 f(x) dx = -\frac{1}{2}.$$

$$F(2) = \int_0^2 f(x) dx = -\frac{1}{2} + \frac{1}{2} = 0.$$

$$F(3) = \int_0^3 f(x)dx = -\frac{1}{2} + \frac{1}{2} + 1 = 1.$$

$$F(4) = \int_0^4 f(x)dx = -\frac{1}{2} + \frac{1}{2} + 2 = 2.$$

$$F(5) = \int_0^5 f(x)dx = -\frac{1}{2} + \frac{1}{2} + 2 + 1 + 2 = 5.$$

Example 2.6. Estimate the definite integral

$$\int_0^1 e^x dx.$$

In this case, we need to use Riemann sums since the area we are trying to determine is not an elementary geometric shape. Therefore using the calculator, using left hand sums with 100 subdivisions, we get

$$\int_0^1 e^x dx \sim 1.71$$