Sections 5.3: The Fundamental Theorem of Calculus

In this section we consider the problem of calculating definite integrals explicitly.

1. The Fundamental Theorem of Calculus Part 1

In Section 4.9 we considered the problem of determining antiderivatives of a given function. In many cases we were able to determine antiderivatives by reversing the operation of differentiation. We saw however that this was not always possible and there were certain functions which we could not determine a formula for an antiderivative. This does not mean that an antiderivative does not exist, and in fact provided $f(x)$ is continuous (except at possibly finitely many points), then we can guarantee that an antiderivative always exists and can be constructed as follows.

Result 1.1. Suppose that f is continuous on $[a, b]$. Then

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of $f(x)$.

Before we talk about why this defines an antiderivative, we make a few observations.

(i) The variable x is one of the limits in the integral - the variable t is a "dummy variable" used to tell us that the function is defined as the **value of an integral**.

(ii) To calculate values of $F(x)$, we need to calculate the integral for a given value of x (which can be done using Riemann sums).

(iii) Since $F(x)$ is an antiderivative of $f(x)$ we have $F'(x) = f(x)$.

(iv) Any other antiderivative of $f(x)$ will be of the form

$$\int_a^x f(t)dt + C$$

where C is an arbitrary constant.

We illustrate with a couple of examples.

Example 1.2. Define

$$Si(x) = \int_0^x \frac{\sin(t)}{t} dt$$

(note that even though $\sin(t)/t$ is not defined at $t = 0$, we can define $Si(x)$ at $x = 0$ to be $Si(0) = 0$).

(i) Fill in the following table for the different values of $Si(x)$.

x	0	1	2	3
$Si(x)$	0	0.937	1.6	1.85

We calculate each of these values using the calculator program with 100 subdivisions and using the left hand sum. Note that we cannot have 0 as our lower limit since $\sin(t)/t$ is undefined at 0, so instead we use 0.01.

(ii) Calculate the derivatives of the following functions:

(a)

$$F(x) = \int_x^0 \frac{\sin(t)}{t} dt.$$

We have

$$F(x) = \int_x^0 \frac{\sin(t)}{t} dt = -\int_0^x \frac{\sin(t)}{t} dt = -Si(x).$$

Thus

$$F'(x) = -S'(x) = -\frac{\sin(x)}{x}$$

by FTC.

(b)

$$F(x) = \int_0^{x^2} \frac{\sin(t)}{t} dt.$$

We have

$$F(x) = \int_0^{x^2} \frac{\sin(t)}{t} dt = Si(x^2).$$

Thus using the chain rule, we have

$$F'(x) = S'(x^2) \cdot 2x = \frac{\sin(x^2)}{x^2} \cdot 2x = \frac{2\sin(x^2)}{x}$$

by FTC.

(c)

$$F(x) = \int_{x^2}^{x^3} \frac{\sin(t)}{t} dt.$$

We have

$$F(x) = \int_{x^3}^{x^2} \frac{\sin(t)}{t} dt = \int_{x^3}^{0} \frac{\sin(t)}{t} dt + \int_0^{x^2} \frac{\sin(t)}{t} dt$$

$$= -\int_0^{x^3} \frac{\sin(t)}{t} dt + \int_0^{x^2} \frac{\sin(t)}{t} dt = -Si(x^3) + Si(x^2).$$

Thus using the chain rule, we have

$$F'(x) = -S'(x^3) \cdot 3x^2 + Si'(x^2) \cdot 2x = -\frac{\sin(x^3)}{x^3} \cdot 3x^2 + \frac{\sin(x^2)}{x^2} \cdot 2x$$

$$= -\frac{3\sin(x^3)}{x} + \frac{2\sin(x^2)}{x}$$

by FTC.

Example 1.3. If

$$F(x) = \int_2^{\sin(x)} \frac{t^2}{e^t} dt$$

deteremine $F'(x)$.

Define

$$G(u) = \int_2^u \frac{t^2}{e^t} dt.$$

Then we have $F(x) = G(\sin(x))$. Therefore, using the chain rule, we have

$$F'(x) = G'(\sin(x)) \cdot \cos(x) = \frac{(\sin(x))^2}{e^{\sin(x)}} \cdot \cos(x).$$

Before we move on, we make an important observation regarding FTC Part 1.

Remark 1.4. FTC Part 1 is extremely important because it guarantees the existence of antiderivatives of any continuous function, and even more, it explains exactly how to calculate the values of an antiderivative. The reason this is important is because there are many functions for which we **cannot** write down an antiderivative (algebraically), but FTC Part 1 does allow us to find an antiderivative (even though it cannot be written down as an expression).

2. Fundamental Theorem of Calculus Part 2

Recall that our original goal was to determine a way to calculate definite integrals exactly. FTC Part 1 guarantees the existence of an antiderivative, but it does not tell us how to calculate definite integrals exactly (indeed, it relies upon being able to calculate a definite integral). We can however use FTC Part 1 to determine a way to calculate definite integrals exactly.

Result 2.1. (Fundamental Theorem of Calculus Part 2) If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. This is a simple consequence of FTC Part 1. Specifically, we know

$$G(x) = \int_a^x f(t)dt$$

is an antiderivative of $f(x)$. Therefore, if $F(x)$ is any other antiderivative, then $F(x) = G(x) + C$ for some constant C. Then

$$F(b) - F(a) = G(b) + C - (G(a) + C) = G(b) - G(a)$$

$$= \int_b^a f(t)dt - \int_a^a f(t)dt = \int_a^b f(t)dt.$$

Specifically,

$$F(b) - F(a) = \int_a^b f(x)dx.$$

□

This means to calculate the definite integral of a function $f(x)$ over an interval $[a, b]$, we do the following:

(i) Find some antiderivative $F(x)$.
(ii) Take the difference $F(b) - F(a)$ which is equal to the definite integral.

Thus the problem of determining the exact value of a definite integral is equivalent to finding an antiderivative and evaluating it at appropriate points. We finish by illustrating FTC Part 2 with a couple of examples.

Example 2.2. Evaluate the following definite integrals exactly.

(i)

$$\int_0^{\frac{\pi}{2}} \sin(x)dx.$$

An antiderivative of $\sin(x)$ is $-\cos(x)$. Therefore, using FTC Part 2, we have

$$\int_0^{\frac{\pi}{2}} \sin(x)dx = -\cos(x)\Big|_0^{\pi/2} = -\cos(\frac{\pi}{2}) - (-\cos(0)) = 1.$$

(ii)

$$\int_0^1 e^x dx.$$

An antiderivative of e^x is e^x. Therefore, using FTC Part 2, we have

$$\int_0^1 e^x dx = e^x\Big|_0^1 = e - e^0 = e - 1.$$

(iii)

$$\int_0^2 (x^2 - 2x)dx.$$

An antiderivative of $x^2 - 2x$ is $x^3/3 - x^2$. Therefore, using FTC Part 2, we have

$$\int_0^2 (x^2 - 2x)dx = \frac{x^3}{3} - x^2\Big|_0^2 = (\frac{8}{3} - 4) - 0 = -\frac{4}{3}.$$

Example 2.3. Is

$$\int_{-1}^1 \frac{1}{x}dx = 0?$$

No - we cannot apply FTC Part 2 since the function $f(x) = 1/x$ is not continuous on the interval $[-1, 1]$.